Delay-dependent stability analysis of neutral systems using positive polynomials optimization

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Abstract: The main goal of this paper is to investigate the stability of neutral systems, by using some features of the method introduced by Zhang, Knospe and Tsiotras Zhang et al. (2003) for the standard time-delay case. The central idea consists in replacing the delay by some appropriate Padé approximation, obtaining a (sufficient) delay-dependent stability condition for the neutral system. Such a condition can be rewritten as a positivity constraint on a two-variable matrix polynomial, leading to an effective computation of the delay margin by using sum-of-squares representations of multivariable positive matrix polynomials.

Keywords: neutral systems, Padé approximations, positive polynomials, SOS representations

1. INTRODUCTION

An important problem in asymptotic stability issues of time delay systems consists of finding the set of values of the delay for which the system achieves asymptotic stability. If this set is $[0, \infty]$, then the system exhibits delay independent asymptotic stability. There are many approaches to tackle this problem, for instance in Datko (1978) an analytical solution is given, based on the analysis of a radius function which depends continuously on the delay. A matrix-pencil approach can be found in Niculescu (1998), whereas in Olgac and Sipahi (2004, 2005) a direct method is proposed based on the involved analysis of clusters of infinite roots given by the characteristic equation associated to the system. Another way to treat this problem stems from robustness concepts, Niculescu et al. (1995, 1997); Zhang et al. (2003). In Zhang et al. (2003) the problem is turned into one of robust stability analysis of a linear (comparison) system, free of delays, but with uncertain parameters, using "covering" and "under-covering" sets of the delay element $e^{-\tau s}$. The approach relies on exploiting some remarkable properties of the Padé approximation of $e^{-\tau s}$. The robust stability of the system created using the outer covering set turns into a sufficient condition for the asymptotic stability of the original system. The result obtained in this way has a very low degree of conservatism, with an upper bound depending only on the order of the Padé approximation, because the robust stability of the system created using the inner covering set gives a necessary asymptotic stability condition. Moreover, from numerical point of view, the conditions yield an algorithm which is based on the test of a *finite set of LMI*, which can be implemented using e.g. the LMI Toolbox, Gahinet et al. (1995). In the multiple delay case, the number of LMI is expected to depend exponentially on the number of delays of the system.

The goal of this paper is to extend this approach to the case of neutral systems (useful for instance in the stability analysis performed in Wang et al. (2002)). In Section 2 we will formulate the problem together with some analytical properties peculiar to neutral systems. Using the work of Datko (1978), we give an analytical stability criterion for the case of neutral systems. By exploiting this criterion in combination with the aforementioned "covering" set of the delay element $e^{-\tau s}$, in Section 3 we transform Datko's stability condition into a positivity test on the set of two-variable matrix polynomials, which allows us to propose a numerically tractable optimization problem for performing this test. In Section 4 we present our numerical results, showing how the sum-of-squares representation in Section 3 is implemented for an efficient computation of the stability delay margin. The paper is completed by drawing some conclusions and looking at future research directions concerning our work.

Some other extensions of the standard delay case to the neutral situation have been already proposed by Ionescu and Ştefan (2007), following closely the procedure Zhang et al. (2003) and obtaining a more involved set of LMI. In a similar line with us, Peet et al. (2008) proposed recently a delay margin stability test involving also sum-of-squares representations of multivariable positive polynomials.

2. PROBLEM FORMULATION

Consider a neutral system with a single delay:

$$\dot{x}(t) - B\dot{x}(t-\tau) = A_0 x(t) + A_1 x(t-\tau). \qquad (1)$$

Its characteristic equation is given by:

$$\det(G_\tau(s)) = 0, \qquad (2)$$

where $G_\tau(s) = s\,(I - Be^{-s\tau}) - A_0 - A_1 e^{-s\tau}$. Define the spectral limit of $G_\tau(s)$ as

$$\sigma(G_\tau) = \sup\{Re\ s \mid \det(G_\tau(s)) = 0\},$$

if $\det(G_\tau(s))$ not identical 0. If $\det(G_\tau(s)) = 0$ for every $s \in \mathbb{C}$, then, by definition, $\sigma(G_\tau) = -\infty$.

The goal is to determine how the stability properties of (1) depend on a change in the delay τ. In particular, if the neutral system (1) is asymptotically stable for $\tau = 0$, we want to find (or at least to approximate) the largest value of τ that preserves this property. In other words, it all boils down to finding the largest delay interval $[0, \overline{\tau})$, $\overline{\tau} > 0$ where the system (1) is asymptotically stable.

Subsequently, we present some properties of the function $\sigma(G_\tau)$. Necessary and sufficient conditions for asymptotic stability - actually exponential stability - have been obtained in Datko (1978), based on the continuity of the spectral limit as a real function of the variable τ.
Consider the analytical functions $A_\tau(s) = A_0 + A_1 e^{-s\tau}$ and $B_\tau(s) = I - Be^{-s\tau}$. One can immediately see that

$$G_\tau(s) = sB_\tau(s) - A_\tau(s). \tag{3}$$

The results below - Theorems 1, 2 and 3 - are practically special cases (single delay situation) of Theorems 2.3, 3.2 and 3.4 in Datko (1978).

Theorem 1. The function $G_\tau(s)$ has a spectral limit function $\sigma(G_\tau)$ which is continuous on $(0, \infty)$. If, in addition, B is a Schur matrix (all eigenvalues belong to the unit disc), then $\sigma(G_\tau)$ is continuous on $[0, \infty)$.

The following results give necessary and sufficient exponential stability conditions that will provide an analytic method for determining the largest delay interval for which (1) remains asymptotically stable.
The method consists in deciding whether the system (1) is asymptotically stable for given τ_0. It is known from the general theory of delayed and neutral differential equations Hale (1977), that, if $\sigma(G_{\tau_0}) < 0$, then the system (1) is asymptotically stable for τ_0. Thus, if $\sigma(G_0) < 0$ and $\tau_0 \in [0, \overline{\tau})$, where $\overline{\tau}$ is the smallest number for which $\sigma(G_{\overline{\tau}}) = 0$, it follows by the continuity property in Theorem 1 that $\sigma(G_{\tau_0}) < 0$ and therefore the system (1) is asymptotically stable for τ_0.

Theorem 2. Let the system (1) be under the assumption that the following conditions are satisfied:
(i) B is a Schur matrix;
(ii) $\sigma(G_0) < 0$.

Then the following statements all hold:
1. There exists a maximal interval $[0, \overline{\tau})$, $\overline{\tau} > 0$ such that $\sigma(G_\tau) < 0$, $(\forall)\ \tau \in [0, \overline{\tau})$, hence for all τ in this interval the system (1) is exponentially stable.
2. Moreover, if $\overline{\tau} < \infty$, then $\sigma(G_{\overline{\tau}}) = 0$ and the system has a periodic solution for $\tau = \overline{\tau}$.

From this, the next stability criterion follows:

Theorem 3. If the system (1) satisfies the following:
1. B is a Schur matrix;
2. $B_0^{-1}(s)A_0(s)$ is Hurwitz;
3. $\det G_\tau(j\omega) \neq 0$, for all $\omega \geq 0$, $\tau \in [0, \overline{\tau}]$, then the system (1) is exponentially stable for all $\tau \in [0, \overline{\tau}]$.

A practical consequence of Theorem 3 is that $\det G_\tau(j\omega) \neq 0$ becomes an essential condition in determining the largest interval $[0, \overline{\tau})$ that ensures asymptotic stability. As in Peet et al. (2008), the computational approach to this condition is based on a result by Zhang et al. (2003).

Consider the Padé approximation of the delay element $e^{-\tau s}$

$$R_m(s) = \frac{D_m(s)}{D_m(-s)}, \tag{4}$$

where

$$D_m(s) = \sum_{k=0}^{m} c_k s^k,\ c_k = (-1)^k \frac{(2m-k)!m!}{(2m)!k!(m-k)!}.$$

One can introduce the following value sets

$$\Omega_d(\omega, \overline{\tau}) = \left\{e^{-j\omega\tau} \mid \tau \in [0, \overline{\tau}]\right\},$$
$$\Omega_o(\omega, \overline{\tau}) = \left\{R_m(j\omega\alpha_m\tau) \mid \tau \in [0, \overline{\tau}]\right\},$$
$$\Omega_i(\omega, \overline{\tau}) = \left\{R_m(j\omega\tau) \mid \tau \in [0, \overline{\tau}]\right\}$$

where $\alpha_m = \dfrac{\omega_{cm}}{2\pi}$ and ω_{cm} is the smallest frequency where the phase has the value -2π, i.e.: $\omega_{cm} = \min\{\omega > 0 \mid R_m(j\omega) = 1\}$. According to Lemma 6 in Zhang et al. (2003) (which uses results from Lam (1990)), we have the following remarkable properties:

1. All poles of $R_m(s)$ are in the open left complex half plane.
2. For any $\overline{\tau} > 0$ and any $\omega > 0$, one has $\Omega_i(\omega, \overline{\tau}) \subseteq \Omega_d(\omega, \overline{\tau}) \subseteq \Omega_o(\omega, \overline{\tau})$ and $\lim\limits_{m\to\infty} \alpha_m = 1$.

Thus, Ω_o and Ω_i are the "covering" and the "under-covering" value sets, respectively. Using $R_m(j\alpha_m\omega\tau)$ instead of $e^{-j\omega\tau}$ in the expression of $G_\tau(j\omega)$, the condition 3 of Theorem 3 becomes only sufficient for stability (the gap to necessity becoming smaller as m grows). However, the condition becomes computationally tractable via a sum-of-squares polynomial approach that we detail in the next section.

3. SUM-OF-SQUARES APPROACH

In this section, we derive our main contribution—a polynomial positivity condition that will allow us to determine whether or not, for a given $\overline{\tau}$, the system (1) exhibits asymptotic stability for all $\tau \in [0, \overline{\tau}]$. Like Peet et al. (2008), we use the Padé approximation of a complex exponential to obtain polynomial conditions, but we work with matrix polynomials. Condition 3 of Theorem 3 can be checked by solving the the optimization problem

$$\begin{aligned} \lambda^\star = \max\ & \lambda \\ \text{s.t.}\ & G_\tau(j\omega)G_\tau(j\omega)^H \succeq \lambda I \\ & \forall\omega \geq 0,\ \forall\tau \in [0, \overline{\tau}] \end{aligned} \tag{5}$$

If $\lambda^\star$ is nonzero, then we decide that the system (1) is stable. If $\lambda^\star$ is (numerically equal to) zero, we decide that the system is unstable.

Substituting $R_m(j\alpha_m\omega\tau)$ for $e^{-j\omega\tau}$, the matrix polynomial appearing in the constraint of (5) becomes

$$\begin{aligned} P(\omega,\tau) &\stackrel{\text{def}}{=} G_\tau(j\omega)G_\tau(j\omega)^H \\ &= \omega^2(I + BB^T - B^T R_m(j\alpha_m\omega\tau) - BR_m(-j\alpha_m\omega\tau)) \\ &\quad - j\omega(A_0^T - BA_1^T - BA_0^T R_m(j\alpha_m\omega\tau) + A_1^T R_m(-j\alpha_m\omega\tau)) \\ &\quad + j\omega(A_0 - A_1B^T - A_0B^T R_m(-j\alpha_m\omega\tau) + A_1 R_m(j\alpha_m\omega\tau)) \\ &\quad + A_1A_0^T R_m(j\alpha_m\omega\tau) + A_0A_1^T R_m(-j\alpha_m\omega\tau) \\ &\quad + A_0A_0^T + A_1A_1^T. \end{aligned}$$

We make the substitution $\theta = \omega\tau$, with the purpose to decrease the total degree of the above polynomial. We also denote $E_1(\theta) = D_m(j\alpha_m\theta)D_m(-j\alpha_m\theta)$ and $E_2(\theta) =$

$D_m(j\alpha_m\theta)^2$. (We keep in mind that these polynomials depend on the degree of approximation m.) By eliminating the denominator of $P(\omega, \tau)$, the constraint of (5) is replaced with

$$\tilde{P}(\theta, \tau) - \lambda\tau^2 E_1(\theta) I \succeq 0, \ \forall\theta \geq 0, \ \ \forall\tau \in [0, \overline{\tau}], \qquad (6)$$

where

$$\begin{aligned}\tilde{P}(\theta, \tau) &= \theta^2[(I + BB^T)E_1(\theta) - B^T E_2(\theta) - BE_2(-\theta)] \\ &- j\omega[(A_0^T - BA_1^T)E_1(\theta) - BA_0^T E_2(\theta) + A_1^T E_2(-\theta)] \\ &+ j\omega[(A_0 - A_1B^T)E_1(\theta) - A_0B^T E_2(-\theta) + A_1E_2(\theta)] \\ &+ A_1A_0^T E_2(\theta) + A_0A_1^T E_2(-\theta) + (A_0A_0^T + A_1A_1^T)E_1(\theta).\end{aligned}$$

This polynomial depends on two variables, θ and τ, both real. We can transform it into a hybrid polynomial $\tilde{P}(\theta, z)$ (see Dumitrescu and Ştefan (2008) for definitions and theory), since $\tau \in [0, \overline{\tau}]$, with the substitution

$$\tau = \left(1 + \frac{z + z^{-1}}{2}\right)\frac{\overline{\tau}}{2}, \qquad (7)$$

where z in on the unit circle $\mathbb{T}$. Let also denote $E(\theta, z) = \tau^2 E_1(\theta)$, with τ replaced as in (7). We have obtained the following approximation of (5)

$$\begin{aligned}\lambda_0 = &\max \ \lambda \\ &\text{s.t.} \ \ \tilde{P}(\theta, z) - \lambda E(\theta, z)I \succeq 0 \\ &\qquad \forall\omega \geq 0, \ \forall z \in \mathbb{T}\end{aligned} \qquad (8)$$

Due to the "covering" effect of the Padé approximation, we obtain $\lambda_0 \leq \lambda^\star$.

To be able to solve (8), we appeal to a sum-of-squares formulation. The hybrid polynomial (with matrix coefficients) $\tilde{P}(\theta, z) - \lambda E(\theta, z)$ is positive semidefinite for all $\theta \geq 0$, $z \in \mathbb{T}$, if there exist sum-of-squares polynomials $S_0(\theta, z)$, $S_1(\theta, z)$ such that

$$\tilde{P}(\theta, z) - \lambda E(\theta, z)I = S_0(\theta, z) + \theta S_1(\theta, z). \qquad (9)$$

Note that a sum-of-squares hybrid polynomial can be written as

$$S(\theta, z) = \sum_{\ell=1}^{\nu} H_\ell(t, z)H_\ell^*(t, z^{-1}). \qquad (10)$$

The expression (9) is similar to those discussed in Scherer and Hol (2006); since θ belongs to an unbounded interval, necessity cannot be stated. The advantage of our approach, in particular of the substitution (7), is that the bounded character of the unit circle permits an expression with less sum-of-squares (there are two in (9)) than in the method of Peet et al. (2008) (where four sum-of-squares are used in a similar positivity condition, but in a different setup, where the Positivstellensatz is used). Let us also note that the degree of the sum-of-squares may be arbitrarily high. In practice, we have to bound the degrees. However, from our experience with sum-of-squares parameterizations, we can affirm that taking the minimal degree (that of the parameterized polynomial) gives almost optimal results. This minimal degree is $(2m + 2, 2)$; that the degree in θ is $2m + 2$ is visible from the expression of $\tilde{P}(\theta, z)$; the degree in z is 2, due to (7) and the presence of the term τ^2 in (7). These degrees are smaller than those used by Peet et al. (2008), for 2×2 matrices; note also that the degrees of our polynomials do not depend on the size of the matrices A_0, A_1, B; for larger matrices, our test gains a clear complexity advantage over the test in Peet et al. (2008), where $\det G_\tau(\omega)$ is computed and hence the degrees of the sum-of-squares grow with the size of the matrices.

We end up with the optimization problem

$$\begin{aligned}\lambda_1 = &\max \ \lambda \\ &\text{s.t.} \ \ (9)\end{aligned} \qquad (11)$$

This problem can be solved using the LMI equivalents of hybrid sum-of-squares, see Dumitrescu and Ştefan (2008), which transform it into an SDP problem, in a way similar to that for problems with real or trigonometric sum-of-squares polynomials. Due to the bounded degrees of the sum-of-squares, we have $\lambda_1 \leq \lambda_0 \leq \lambda^\star$. We have thus obtained a computational version of our sufficient stability test.

4. NUMERICAL EXAMPLE

We have implemented the SDP problem equivalent to (11) using the SDP library SeDuMi by Sturm (1999). We take the degree of $S_0(\theta, z)$ equal to the degree of the polynomial on the left of (9), namely $(2m + 2, 2)$; accordingly, the degree of $S_1(\theta, z)$ is $(2m, 2)$. For a degree of Padé approximation $m = 3, 4, 5$, the total degrees of the polynomials used in the sum-of-squares approach of Peet et al. (2008) are 14, 18 and 22, respectively. In our case, the total degree is $2m+4$ and has the values 10, 12 and 14, respectively. The execution time necessary to solve (11) is about 3 seconds for $m = 5$.

We apply the test for the example system discussed in Peet et al. (2008) and other previous papers, e.g. Han (2005), defined by matrices

$$A_0 = \begin{bmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{bmatrix}, \ \ A_1 = \begin{bmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{bmatrix}, \quad B = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{bmatrix}.$$

The values λ_1 computed with the SDP equivalent of (11) are shown in Figure 1, for $m = 3, 4, 5$. To decide that the system is stable or not, we must decide if λ_1 is not zero; it is clear from the curves that the decision must be made using as reference the almost constant values obtained for $\overline{\tau}$ greater than a stability threshold. For $m = 3$, the threshold is about 10^{-7} and we can estimate that the maximum value of $\overline{\tau}$ for which the system is stable is approximately 1.804. As m grows, the threshold also grows. The maximum values of $\overline{\tau}$ that ensure stability are 2.153 for $m = 4$ and 2.206 for $m = 5$. These values are comparable to those from Peet et al. (2008) and for $m = 5$ to those obtained with time domain methods like Han (2005). For this example, it can be proved analytically that the maximum admissible value of $\overline{\tau}$ is 2.2255. To see how tight are our results, the solid line curve in Figure 1 represent an upper bound of $\lambda^\star$, denoted $\overline{\lambda}^\star$, which is the minimum value of the minimimum eigenvalue of the matrix polynomial $P(\omega, \tau) = G_\tau(j\omega)G_\tau(j\omega)^H$ that appears in (5), computed on a grid of points. We remark that the values of λ_1 for $m = 5$ follow closely the values of $\overline{\lambda}^\star$.

5. CONCLUSION

The aim of this paper is to extend the approach in Zhang et al. (2003) to the case of neutral systems, using instead

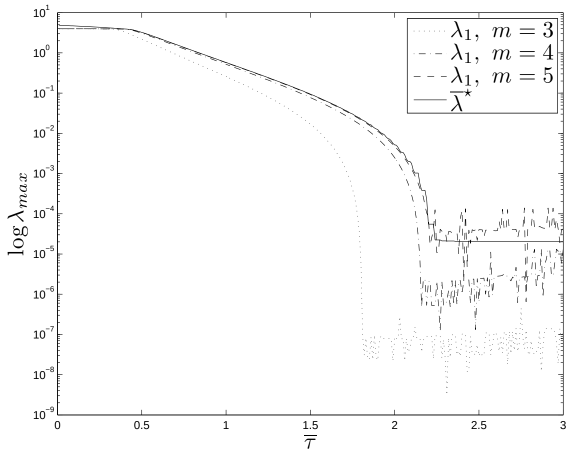

Fig. 1. Values of the solution of (11) for the example.

of standard LMI tests optimization procedures on SOS representations of two-variable positive polynomials. The presence of the delay in the left hand term of the dynamic equation of such kind of systems makes the stability analysis more involved. In a certain context, established by natural assumptions which lead to a well-posed problem (stability for the delay-free system, continuity of the spectral limit function), we have shown that it is possible to follow the framework in Zhang et al. (2003), using an appropriate Padé approximation of the delay element. The result is analogous to the one obtained by Peet et al. (2008), but we differentiate ourselves by using a straightforward reformulation of the stability condition in terms of positivity of matrix polynomials, which gives a lower complexity sum-of-squares test.

Recently, several methods have been proposed in the literature for the robustness analysis of arbitrary Hurwitz quasi-polynomials with respect to the time-delays (see for instance Morarescu (2006)). It would be interesting to combine the outcome of the different methods, in order to reach less conservative delay stability margins or to guarantee a certain degree of conservatism for situations of practical importance.

The extension to the multiple delay case can use a similar framework. Since for each delay there will be an extra variable, the complexity of our sum-of-squares approach will increase. However, we appreciate that problems with 2-3 delays can still be solved practically.

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