LEAST SQUARES DESIGN OF THREE-DIMENSIONAL FILTER BANKS USING TRANSFORMATION OF VARIABLES

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ABSTRACT

The topic discussed here is the design of three-dimensional two-channel filter banks with quincunx sampling. The technique used is the transformation of variables, where we reduce the original problem to designing filters with minimum stopband energy. The filters are designed using a least squares manner. To prove the capability of our method we compare our results with another approach of the problem.

Index Terms— three-dimensional filter banks, transformation of variables, stopband energy, minimization

1. INTRODUCTION

The purpose of this paper is to design two-channel three-dimensional (3-D) FIR filter banks (FBs), which have a truncated octahedron (TRO) ideal passband shape. The problem has been addressed using transformation of variables [1, 2] (using parametric Bernstein polynomials and the window method, respectively) or interior point methods [3]. Here, we propose a least-squares approach, exploiting the concept of transformation of variables in a new way. Our method has three steps. In the first, the transformation is optimized without considering the properties of the FB final filter. The second step is iterative and attempts the stopband energy optimization via an heuristic that makes the optimization problem convex. The optional third step is a nonlinear optimization of the 1-D filters used to build the 3-D filters. Our approach allows the simple introduction of regularity constraints on the filters.

The remainder of this paper is organized as follows. We present in Section 2 the general context for the design of the 3-D filter bank. Next, in Section 3 we introduce a new approach for the design of the transformation function. We tackle in Section 5 the problem of imposing the regularity conditions. We show in Section 6 the result obtained using the proposed method. We conclude in Section 7.

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The notation is standard. Multivariate entities (vectors, matrices) are denoted by bold characters. $\boldsymbol{M}^T$ is the transposed of matrix $\boldsymbol{M}$. $\mathbb{Z}_+$ and $\mathbb{T}$ are the sets of nonnegative integers and complex numbers on the unit circle, respectively. We associate to a trigonometric polynomial $X(\boldsymbol{z})$ [4] the vector $\boldsymbol{x}$ which contains the coefficients of the filter in a halfspace. We denote by Toep$(y_0, y_1, \ldots, y_n)$ the Toeplitz matrix having y_k on the k-th diagonal and by $||\boldsymbol{y}||$ the 2-norm of the vector $\boldsymbol{y}$.

2. THE FILTER BANK

We aim to design FBs like the one from Figure 1. The filters are 3-D real symmetric trigonometric polynomials

$$H(\boldsymbol{z}) = \sum_{\boldsymbol{k}=-\boldsymbol{n}}^{\boldsymbol{n}} h_{\boldsymbol{k}} \boldsymbol{z}^{-\boldsymbol{k}}, \quad h_{\boldsymbol{k}} = h_{-\boldsymbol{k}}, \tag{1}$$

where $\boldsymbol{n} = (n_1, n_2, n_3) \in \mathbb{Z}_+^3$ and $\boldsymbol{z} = (z_1, z_2, z_3) \in \mathbb{T}^3$. The sampling is done on the face-centered orthorhombic (FCO) lattice [5] using the sampling matrix

$$\boldsymbol{D} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}. \tag{2}$$

The aliasing function of the FB is canceled by choosing the following highpass filters:

$$\begin{aligned} H_1(z_1, z_2, z_3) &= z_1^{-K_1} z_2^{-K_2} z_3^{-K_3} F_0(-z_1, -z_2, -z_3) \\ F_1(z_1, z_2, z_3) &= z_1^{K_1} z_2^{K_2} z_3^{K_3} H_0(-z_1, -z_2, -z_3) \end{aligned} \tag{3}$$

with $K_1 + K_2 + K_3$ being odd. The lowpass filters are obtained using the transformation of variables technique [6], as

$$\begin{aligned} H_0(z_1, z_2, z_3) &= H_T(M(\boldsymbol{z})) \\ F_0(z_1, z_2, z_3) &= F_T(M(\boldsymbol{z})) \end{aligned} \tag{4}$$

where $H_T(Z)$ and $F_T(Z)$ are 1-D filters in the variable Z and $M(\boldsymbol{z})$ is a 3-D filter as in (1) with

$$m_{\boldsymbol{k}} = 0, \quad \text{if } k_1 + k_2 + k_3 \text{ is even.} \tag{5}$$

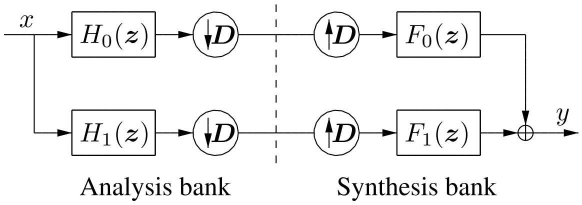


Fig. 1. Two-channel filter bank.

Denoting $D_T(Z) = H_T(Z)F_T(Z)$, the perfect reconstruction condition is imposed on the FB by requiring that

$$D_T(Z) + D_T(-Z) = 1. \tag{6}$$

3. THE TRANSFORMATION FUNCTION

We focus now on designing the transformation function. To quantify the quality of a filter, we use the stopband energy, defined for a filter $H(\boldsymbol{z})$ as

$$E_s(H) = \int_{V_s} H(\boldsymbol{\omega})\, d\boldsymbol{\omega}, \tag{7}$$

where V_s is the stopband region. Furthermore (7) is equivalent to

$$E_s(H) = \boldsymbol{h}^T \cdot \boldsymbol{C} \cdot \boldsymbol{h}, \tag{8}$$

where $\boldsymbol{C}$ is a constant positive definite matrix, $\boldsymbol{C} = \boldsymbol{P}^T \bar{\boldsymbol{C}} \boldsymbol{P}$, with

$$\bar{\boldsymbol{C}} = \int_{V_s} \boldsymbol{C}_3(\omega_3) \otimes \boldsymbol{C}_2(\omega_2) \otimes \boldsymbol{C}_1(\omega_1)\, d\boldsymbol{\omega}, \tag{9}$$

$\boldsymbol{C}_i = \mathrm{Toep}(\mathrm{e}^{j\omega_i n_i}, \ldots, 1, \ldots, \mathrm{e}^{-j\omega_i n_i})$, $i = 1:3$ and

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{0} & 1 & \boldsymbol{0} \\ \boldsymbol{J} & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}^T, \tag{10}$$

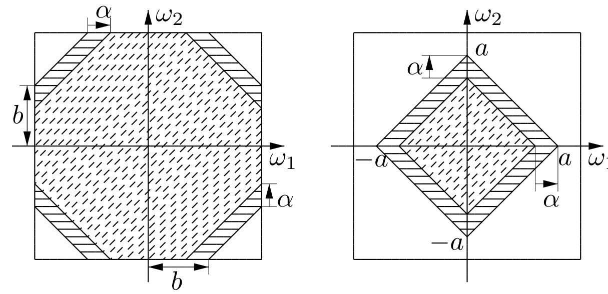
where $\boldsymbol{J}$ is a counteridentity matrix of appropriate dimensions. Thus the computation of the matrix $\boldsymbol{C}$ essentially reduces to solving the integral

$$I = \int_{V_s} \cos\left(k_1\omega_1 + k_2\omega_2 + k_3\omega_3\right) d\boldsymbol{\omega}. \tag{11}$$

The stopband size is determined by a parameter α, as shown in Figure 2, where there are two 2-D sections obtained for constant ω_3 and

$$\begin{array}{lclr} a & = & 3\pi/2 + \alpha - |\omega_3|, & \pi/2 + \alpha \le |\omega_3| \le \pi \\ b & = & \pi/2 + \alpha - |\omega_3|, & 0 \le |\omega_3| \le \pi/2 + \alpha. \end{array} \tag{12}$$

The left figure corresponds to "low" ω_3, in which case the ideal passband and the transition band (dashed) form an octagon. The right figure is for "high" ω_3, where the passband and transition band shape is a rhombus. In both cases, the stopband is represented in white. Due to space limitations, we omit the formulas for (11) on the stopband described in Figure 2. See [7] for the formulas over the ideal passband region ($\alpha = 0$).



(a) Octagon. (b) Rhombus.

Fig. 2. Passband sections through the ω_3 axis.

In a first step, we optimize the transformation $M(\boldsymbol{z})$ by using only information on the typical choices of 1-D polynomials $H_T(Z)$ and $F_T(Z)$, whose values grow from 0 to 1 as Z goes from 0 to 1. Hence, the polynomial $\tilde{M}(\boldsymbol{z}) = 1 + M(\boldsymbol{z})$ should be nearly equal to zero in the stopband, which leads to the design of the transformation by solving the following optimization problem (which takes (5) and (7) into account)

$$\begin{array}{ll} \min\limits_{\tilde{\boldsymbol{m}}} & \tilde{\boldsymbol{m}}^T \boldsymbol{C} \tilde{\boldsymbol{m}} \\ \text{s.t.} & \tilde{m}_{k_1,k_2,k_3} = \delta_{k_1 k_2 k_3}, \quad \text{if } k_1 + k_2 + k_3 = \text{even} \end{array} \tag{13}$$

where δ is the Kronecker symbol. This is the minimization of a convex quadratic subject to linear constraints. Denoting the constraint as $\boldsymbol{\Omega}\tilde{\boldsymbol{m}} = \boldsymbol{\gamma}$ the solution of (13) is

$$\tilde{\boldsymbol{m}} \;=\; -\tfrac{1}{2}\boldsymbol{C}^{-1}\boldsymbol{\Omega}^T\boldsymbol{\tau}, \tag{14}$$

with $\boldsymbol{\Omega}\boldsymbol{C}^{-1}\boldsymbol{\Omega}^T\boldsymbol{\tau} = -2\boldsymbol{\gamma}$.

In the second step, we aim to attain the actual optimization purpose, which is the minimization of the stopband energy of the filters (4), for given 1-D filters $H_T(Z)$ and $F_T(Z)$. Hence, we try to solve the problem

$$\min_{M(\boldsymbol{z})} \quad \lambda E_s(H_T(M(\boldsymbol{z}))) + (1-\lambda)E_s(F_T(M(\boldsymbol{z}))), \tag{15}$$

where $\lambda \in [0, 1]$ is given. For the purpose of illustration, we consider $H_T(Z)$ and $F_T(Z)$ of degrees two and three, respectively (generalization is immediate), i.e. we have

$$\begin{array}{rcl} H_0(\boldsymbol{z}) & = & a_0 + a_1 M(\boldsymbol{z}) + a_2 M(\boldsymbol{z})^2 \\ F_0(\boldsymbol{z}) & = & b_0 + b_1 M(\boldsymbol{z}) + b_2 M(\boldsymbol{z})^2 + b_3 M(\boldsymbol{z})^3 \end{array} \tag{16}$$

The stopband energies that appear in (15) are nonconvex functions of the transformation $M(\boldsymbol{z})$. To make the problem tractable, we employ the following trick. Let $M_0(\boldsymbol{z})$ be the transformation designed using (13). Since this should be a relatively good approximation of the optimal transformation, we optimize instead of (16) the filters

$$\begin{array}{rcl}\tilde{H}_0(\boldsymbol{z}) & = & a_0 + a_1 M(\boldsymbol{z}) + a_2 M_0(\boldsymbol{z}) M(\boldsymbol{z}) \\ \tilde{F}_0(\boldsymbol{z}) & = & b_0 + b_1 M(\boldsymbol{z}) + b_2 M_0(\boldsymbol{z}) M(\boldsymbol{z}) + \\ & & + b_3 M_0(\boldsymbol{z})^2 M(\boldsymbol{z})\end{array} \tag{17}$$

We obtain the optimization problem

$$\min_{M(\boldsymbol{z})} \quad \lambda E_s(\tilde{H}_0(\boldsymbol{z})) + (1-\lambda) E_s(\tilde{F}_0(\boldsymbol{z})). \tag{18}$$

This is a convex quadratic in the coefficients of $M(\boldsymbol{z})$. Indeed, we rewrite (17) as

$$\begin{array}{rcl}\tilde{\boldsymbol{h}}_0 & = & \Phi_1 \boldsymbol{m} + \Psi_1 \\ \tilde{\boldsymbol{f}}_0 & = & \Phi_2 \boldsymbol{m} + \Psi_2\end{array} \tag{19}$$

where Φ_1 and Φ_2 are two corresponding convolution matrices and Ψ_1 and Ψ_2 denote the free terms, respectively. Using (19), the problem (18) now becomes

$$\min_{\boldsymbol{m}} \quad \boldsymbol{m}^T \boldsymbol{\Theta} \boldsymbol{m} + 2\boldsymbol{\eta}\boldsymbol{m} \tag{20}$$

where $\boldsymbol{\Theta} = \Phi_1^T \tilde{\boldsymbol{C}} \Phi_1 + \Phi_2^T \breve{\boldsymbol{C}} \Phi_2$ and $\boldsymbol{\eta} = \Psi_1^T \tilde{\boldsymbol{C}} \Phi_1 + \Psi_2^T \breve{\boldsymbol{C}} \Phi_2$, with $\tilde{\boldsymbol{C}}$ and $\breve{\boldsymbol{C}}$ being two positive semidefinite matrices according to (8). The unique solution of the convex optimization problem is

$$\boldsymbol{m} \quad = \quad -\boldsymbol{\Theta}^{-1} \boldsymbol{\eta}^T. \tag{21}$$

The problem (18) can be solved repeatedly, each time initializing $M_0(\boldsymbol{z})$ with the previous solution. The iterations are repeated as long as the criterion (15) decreases.

4. OPTIMIZATION OF 1-D FILTERS

We propose now a method that can further improve our results. Considering $M_1(\boldsymbol{z})$ the solution of the problem (18) we construct the optimization problem

$$\begin{array}{ll}\min\limits_{\boldsymbol{a},\boldsymbol{b}} & \lambda E_s(H_T(M_1(\boldsymbol{z}))) + (1-\lambda) E_s(F_T(M_1(\boldsymbol{z}))) \\ \text{s.t.} & (6)\end{array} \tag{22}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are coefficients of the 1-D filters $H_T(Z)$ and $F_T(Z)$, respectively.

5. THE REGULARITY CONDITIONS

Regularity conditions can be added easily to the above optimization problems. According to e.g. [2] the regularity conditions have to be imposed on both the 1-D filters $H_T(Z)$ and $F_T(Z)$ and on the transformation function $M(\boldsymbol{z})$. We describe here the latter case.

For an order S of zero derivatives the conditions on the filter $M(\boldsymbol{z})$ are

$$\left.\frac{\partial^L M(\omega_1, \omega_2, \omega_3)}{\partial \omega_1^\ell \partial \omega_2^o \partial \omega_3^p}\right|_{(\omega_1,\omega_2,\omega_3)=(\pi,\pi,\pi)} = 0, \quad \forall L = 1:S, \tag{23}$$

$\forall \ell, o, p, \ell+o+p = L$. We also require that $M(\pi,\pi,\pi) = -1$. To design a transformation function we embed the equality constraints over $M(\boldsymbol{z})$ in the optimization problem (20). Thus we obtain the problem

$$\begin{array}{ll}\min\limits_{\boldsymbol{m}} & \boldsymbol{m}^T \boldsymbol{\Theta} \boldsymbol{m} + 2\boldsymbol{\eta}\boldsymbol{m} \\ \text{s.t.} & \boldsymbol{\Upsilon} \boldsymbol{m} = \boldsymbol{\zeta}\end{array} \tag{24}$$

The solution of the problem (24) is

$$\boldsymbol{m} \quad = \quad -\boldsymbol{\Theta}^{-1}\boldsymbol{\eta}^T - \tfrac{1}{2}\boldsymbol{\Theta}^{-1}\boldsymbol{\Upsilon}^T \boldsymbol{\xi}, \tag{25}$$

where $\boldsymbol{\Upsilon}\boldsymbol{\Theta}^{-1}\boldsymbol{\Upsilon}^T\boldsymbol{\xi} = -2\boldsymbol{\Upsilon}\boldsymbol{\Theta}^{-1}\boldsymbol{\eta}^T - 2\boldsymbol{\zeta}$.

6. DESIGN EXAMPLES

We consider designing a filter bank using a transformation function of degree $\boldsymbol{n} = (3,3,3)$, which is a filter of size $7 \times 7 \times 7$. We use the following 1-D filters [8]

$$\begin{array}{rcl}H_T(Z) & = & -\frac{1}{4}(Z+1)(Z-3) \\ F_T(Z) & = & -\frac{1}{12}(Z+1)(Z^2+Z-8).\end{array} \tag{26}$$

To prove the effectiveness of our method we have also designed 3-D filter banks using the window method [2] and compared the stopband energies for the two methods.

We have made tests using $\alpha = [0.1, 0.15, 0.2, 0.25, 0.3]\pi$. For the window method we have used a Kaiser window with the β parameter having values on the grid $0:0.1:3$ and chose the best case (smallest energy) for each value of α. The values obtained using the window method and the methods from (18) and (24) using different value for S, along with the results obtained by solving the problem (22) are listed in the Table 1. (To solve the problem (22) we have used the `fmincon` function from Matlab.) We have considered $\lambda = 0.5$ for all the tests.

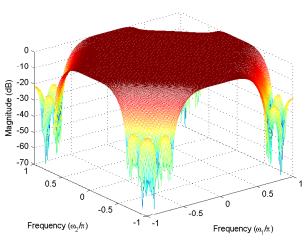
We show in Figure 3 slices of the magnitude response for the $H_0(\boldsymbol{z})$ filter on the ω_3 axis, at $\omega_3 = [0.2, 0.4, 0.6, 0.8]\pi$.

7. CONCLUSIONS

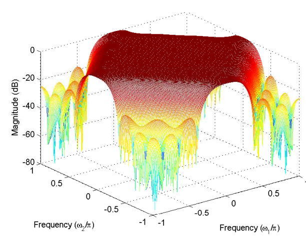
We have proposed a new method for designing 3-D FBs with a TRO passband with the aid of a transformation of variables. Our approach reduces the initial problem to designing filters with minimum stopband energy. Furthermore using our method one can easily introduce regularity constraints in the design problem. Comparing our method with an existent design we have shown that by using our approach filters with better performance (smaller stopband energy) can be obtained.

Table 1. Stopband energies using different algorithms.

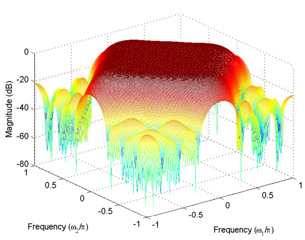
α/π	Window method [2]	Problem (18)	Problem (24), $S = 3$	Problem (24), $S = 5$	Problem (22)
0.1	0.006351	0.006322	0.006330	0.007002	0.005966
0.15	0.002689	0.002662	0.002670	0.003023	0.002481
0.2	0.001117	0.001085	0.001093	0.001256	0.001004
0.25	0.000469	0.000431	0.000437	0.000507	0.000397
0.3	0.000207	0.000166	0.000171	0.000198	0.000153



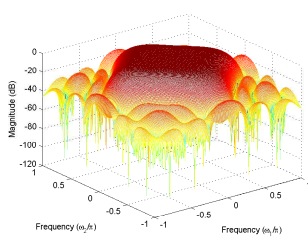
(a) Response at $\omega_3 = 0.2\pi$.



(b) Response at $\omega_3 = 0.4\pi$.



(c) Response at $\omega_3 = 0.6\pi$.



(d) Response at $\omega_3 = 0.8\pi$.

Fig. 3. Magnitude responses for the $H_0(\boldsymbol{z})$ filter at $\omega_3 = [0.2, 0.4, 0.6, 0.8]\pi$.

8. ACKNOWLEDGEMENT

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