MINIMAX DESIGN OF ADJUSTABLE FIR FILTERS USING 2D POLYNOMIAL METHODS

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ABSTRACT

The problem under study here is the minimax design of linear-phase lowpass FIR filters having variable passband width and implemented through a Farrow structure. We have two main contributions. The first is the design of adjustable FIR filters without discretization, using 2D positive trigonometric polynomials, an approach leading to semidefinite programming (SDP) formulation of the design problem. The second is to modify the design problem by a special choice for the passband and stopband edges of the variable FIR filter. The advantage is a lower implementation complexity. The new problem is solved using positive hybrid real-trigonometric polynomials and their SDP parameterization. Design examples prove the viability of our methods.

Index Terms— adjustable FIR filters, positive polynomials, semidefinite programming

1. INTRODUCTION

The problem considered in this paper is the design of adjustable filters defined by

$$H(z,p) = \sum_{k=0}^{K} p^k H_k(z) = \sum_{k=0}^{K} p^k \sum_{n=-N}^{N} h_{k,n} z^{-n}, \qquad (1)$$

where $H_k(z)$, $k = 0 : K$, are symmetric FIR filters of order $2N$ (i.e. $h_{k,n} = h_{k,-n}$), and $p \in \mathbb{R}$ is variable. We have assumed that the filters $H_k(z)$ are zero-phase only for presentation reasons; since only magnitude is interesting, $H_k(z)$ may be any linear-phase type I filters with the same group delay. The implementation of the adjustable filter (1) is made via the Farrow structure [1] shown in Figure 1. The parameter p can be replaced by $p - p_0$, with implementation benefits described in [2]; this can be viewed as a transformation of the coefficients of filters $H_k(z)$ and has no relevance on the design method.

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We want to design lowpass filters of form (1) with adjustable passband and stopband widths. For $p = \theta$, a typical design specification leads to the minimax optimization problem

$$\begin{aligned} \min \quad & \gamma_s \\ \text{s.t.} \quad & 1 - \gamma_p \le H(e^{j\omega}, \theta) \le 1 + \gamma_p,\ \forall \omega \in [0, \theta - \Delta] \\ & -\gamma_s \le H(e^{j\omega}, \theta) \le \gamma_s,\ \forall \omega \in [\theta + \Delta, \pi] \end{aligned} \qquad (2)$$

Here, γ_b is a prescribed passband error bound and the stopband error γ_s is minimized. A feasibility problem could be solved as well, with fixed γ_s. The parameter θ takes values in an interval $[\theta_l, \theta_u]$ and so the passband edge varies between $\theta_l - \Delta$ and $\theta_u - \Delta$. The width of the transition band is constant and equal to 2Δ.

The optimization of adjustable filters was typically performed using a least-squares criterion, see e.g. [3, 4]. Minimax optimization was employed in [2], using linear programming, and in [5], using semidefinite programming (SDP). In both papers, the formulations are obtained through discretization of (2).

Here, we attempt a solution without discretization, using properties of positive bivariate (2D) polynomials. In Section 2, we transform (2) into an optimization problem involving 2D trigonometric polynomials and solved practically optimally using SDP. In Section 3, we modify problem (2) such that the transition band width is no longer constant. The modified problem can be solved using positive hybrid real-trigonometric polynomials. A design example shows the advantages of the modified problem with respect to (2); the complexity of the filter (1), given by the degrees necessary for satisfying prescribed error bounds, is lower for the modified problem.

2. TRIGONOMETRIC POLYNOMIALS APPROACH

The problem (2) can be solved using 2D trigonometric polynomials by changing the significance of the parameter p from (1). Since the parameter θ from (2) has the clear meaning of a frequency, we introduce a new complex variable $\zeta = e^{j\theta}$,

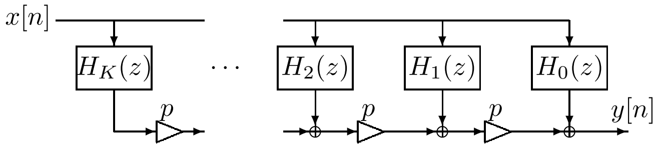

Fig. 1. Implementation of adjustable filters.

taking values on the unit circle, like z. We take the parameter from (1) to be $p = \cos\theta$, which is not a limitation, since $\cos\theta$ takes real values in the interval $[\cos\theta_u, \cos\theta_l]$ (and cos is a monotonic function on $[0, \pi]$). The implementation from Figure 1 is still valid.

There are four similar constraints in (2). We describe explicitly the treatment of one of them, say

$$H(e^{j\omega}, \theta) - \gamma_p + 1 \geq 0, \ \forall\omega \in [0, \theta - \Delta], \ \forall\theta \in [\theta_l, \theta_u], \quad (3)$$

the mechanism being similar for the others. The variables from (3) are constrained to belong to the set

$$\mathcal{D} = \{(\omega, \theta) \in [-\pi, \pi]^2 \mid D_i(e^{j\omega}, e^{j\theta}) \geq 0, \ i = 1, 2\}, \quad (4)$$

where $D_i(z, \zeta)$ are trigonometric polynomials. The first is

$$D_1(z, \zeta) = \tfrac{1}{2}(z + z^{-1} - e^{-j\Delta}\zeta - e^{j\Delta}\zeta^{-1}). \quad (5)$$

Since

$$D_1(e^{j\omega}, e^{j\theta}) = \cos\omega - \cos(\theta - \Delta),$$

it is clear that the polynomial is positive for $\omega \in [0, \theta - \Delta]$. The second polynomial from (4) is

$$D_2(z, \zeta) = d\zeta^{-1} + c + d^*\zeta, \quad (6)$$

with

$$c = -\cos\tfrac{\theta_u - \theta_l}{2}, \ \ d = \tfrac{1}{2}e^{j(\theta_u + \theta_l)/2}, \quad (7)$$

which gives

$$D_2(e^{j\omega}, e^{j\theta}) = \cos(\theta - \tfrac{\theta_u + \theta_l}{2}) - \cos\tfrac{\theta_u - \theta_l}{2}.$$

Hence, the polynomial (6) is nonnegative for $\theta \in [\theta_l, \theta_u]$ and negative for $\theta \in [-\pi, \pi] \setminus [\theta_l, \theta_u]$. Other descriptions of polynomials with the same property are given in [6], [7, Th.1.15]. Note that, although the polynomials (5), (6) have complex coefficients, they have real values on the unit circle, i.e. for $(z, \zeta) \in \mathbb{T}^2$.

We denote $G(z, \zeta) = H(z, \zeta) - \gamma_p + 1$. Note that $H(z, \zeta)$ is obtained immediately in (1) by putting $p = (\zeta + \zeta^{-1})/2$. The inequality (3) can be interpreted as the positivity of $G(e^{j\omega}, e^{j\theta})$ on the set (4). Using a result from [8] regarding the positivity of multivariate trigonometric polynomials, this is equivalent to the existence of sum-of-squares polynomials $S_i(z, \zeta)$, $i = 0 : 2$, such that

$$G(z, \zeta) = S_0(z, \zeta) + D_1(z, \zeta)S_1(z, \zeta) + D_2(z, \zeta)S_2(z, \zeta). \quad (8)$$

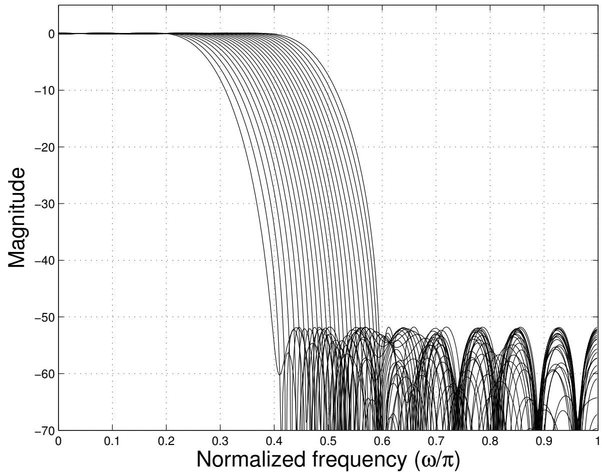

Fig. 2. Frequency responses of adjustable filters designed in Example 1 with $K = 4$, for 25 values of the parameter $\theta \in [0.3\pi, 0.5\pi]$.

Since sum-of-squares polynomials can be parameterized in terms of positive semidefinite matrices [9, 8], it results that (3) is equivalent to a linear matrix inequality (LMI). The same holds for the other constraints of (2), which thus becomes an SDP problem. (Note that similar sum-of-squares results and techniques have been used in [8] for the minimax design of 2D FIR filters.)

The only conservatism appearing at implementation is due to the necessity to impose bounds on the degrees of the sum-of-squares from (8). Practically, we take the sum-of-squares to have minimum degree, i.e. (N, K) for $S_0(z, \zeta)$, $(N - 1, K - 1)$ for $S_1(z, \zeta)$, and $(N, K - 1)$ for $S_2(z, \zeta)$. Experience with 2D filter design and other optimization problems [7] allows us to affirm that the results are practically optimal. We have implemented the solution of (2), via (8), using the SDP library SeDuMi [10].

Example 1. We use the same design data as in [2], namely $\theta_l = 0.3\pi$, $\theta_u = 0.5\pi$, $\Delta = 0.1\pi$, $\gamma_p = 0.01$, and pose the design problem in the same way. For several values K (giving the number of filters $H_k(z)$ in (1), namely $K + 1$), we aim to find minimal orders N such that the optimal stopband attenuation resulted by solving (2) be $\gamma_s \leq 0.00316 = -50$ dB. The results are in the upper half of Table 1. They are similar (but identical only for $K = 4$) to those obtained in [2] using linear programming. For $K > 4$ the order N does not decrease anymore. The number of fixed multipliers needed to implement the adjustable filter as in Figure 1 is $(K + 1)(N + 1)$, shown in the last column of Table 1. Also, K adjustable multipliers are needed. The magnitude responses of the variable filter with $K = 4$ (having the best complexity) are shown in Figure 2. The optimal stopband attenuation is $\gamma_s = 0.00258 = -51.76$ dB.

3. HYBRID POLYNOMIALS APPROACH

We propose now a modification of the design problem (2), that leads to adjustable filters with variable transition band width, having similar performance but lower complexity. The new problem is

$$\begin{aligned} \min \quad & \gamma_s & (9)\\ \text{s.t.} \quad & 1-\gamma_p \le H(e^{j\omega},p) \le 1+\gamma_p,\ \forall \cos\omega \in [p+\tilde{\Delta},1] \\ & -\gamma_s \le H(e^{j\omega},p) \le \gamma_s,\ \forall \cos\omega \in [-1,p-\tilde{\Delta}] \end{aligned}$$

The parameter p keeps the same significance as before, i.e. $p = \cos\theta$, but $\tilde{\Delta}$ is no longer half of the transition band width. The parameter p takes values in the interval $[\tilde{p}_l, \tilde{p}_u]$, with $\tilde{p}_l = \cos\tilde{\theta}_u$, $\tilde{p}_u = \cos\tilde{\theta}_l$. The width of the transition band is $\text{acos}(p-\tilde{\Delta}) - \text{acos}(p+\tilde{\Delta})$ and is no longer constant.

The appealing feature of problem (9) is that the transition band is larger when the passband or the stopband are narrow (and a good stopband attenuation is more difficult to obtain). An example is shown in Figure 3. The solid line parallelogram corresponds to the design in Example 1. Let $\underline{\omega}_b = \theta_l - \Delta$ and $\overline{\omega}_b = \theta_u - \Delta$ be the minimum and maximum values of the passband edge ($\underline{\omega}_b = 0.2\pi$, $\overline{\omega}_b = 0.4\pi$ in our example). A horizontal line through $\omega \in [\underline{\omega}_b, \overline{\omega}_b]$ cuts the parallelogram in two points, whose x-axis coordinates are $\omega_b = \omega$ and $\omega_s = \omega + 2\Delta$, showing the extent of the transition band. The dashed-line curved "parallelogram" corresponds to the design problem (9). (The numerical values are chosen such that the extreme values of the passband edge match the corresponding values from Example 1, i.e. $\text{acos}(\tilde{p}_u + \tilde{\Delta}) = \theta_l - \Delta$, $\text{acos}(\tilde{p}_l + \tilde{\Delta}) = \theta_u - \Delta$, see the design example below.) A horizontal line through $\omega = \text{acos}(p+\tilde{\Delta})$ cuts the curved "parallelogram" in the points whose x-axis coordinates are $\omega_b = \text{acos}(p+\tilde{\Delta})$ and $\omega_s = \text{acos}(p-\tilde{\Delta})$. We see that for small ω the transition band is wider. As ω grows, the transition band becomes narrower. (This is true for the interval $[0, \pi/2]$. For $[\pi/2, \pi]$, the behavior is reversed, i.e. the transition band is wide when the stopband is narrow.)

The solution of (9) is obtained using again the positivity of 2D polynomials, but now these polynomials are hybrid real-trigonometric: the variable p is real, while z is complex. Again, we discuss a single constraint of (9), which can be formulated as follows. The polynomial

$$\tilde{G}(e^{j\omega},p) = H(z,p) - \gamma_p + 1 \tag{10}$$

must be positive on

$$\tilde{\mathcal{D}} = \{(\omega,p) \in [-\pi,\pi] \times \mathbb{R} \mid \tilde{D}_i(e^{j\omega},p) \ge 0,\ i = 1,2\}, \tag{11}$$

with

$$\tilde{D}_1(z,p) = \tfrac{1}{2}(z+z^{-1}) - p - \tilde{\Delta}, \tag{12}$$

$$\tilde{D}_2(z,p) = (p-\tilde{p}_l)(\tilde{p}_u - p). \tag{13}$$

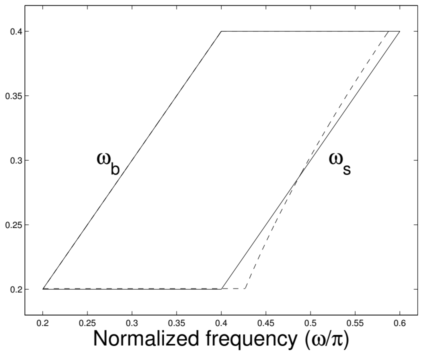

Fig. 3. Transition band $[\omega_b, \omega_s]$ for Example 1, constant transition band width (solid line) and Example 2, variable transition band width (dashed line).

The polynomial (12) describes the extent of the passband and is nonnegative whenever $\cos\omega \ge p + \tilde{\Delta}$. The polynomial (13) is nonnegative for $p \in [\tilde{p}_l, \tilde{p}_u]$ and negative elsewhere.

A result similar to (8) holds for hybrid polynomials when one of the polynomials defining the set (11) is (13) (i.e. explicit bounds are put on the variable p). Some details are in [11] and more will be presented in a forthcoming paper. The polynomial (10) is positive on (11) if there exist sum-of-squares polynomials $\tilde{S}_i(z,p)$, $i = 0:2$, such that

$$\tilde{G}(z,p) = \tilde{S}_0(z,p) + \tilde{D}_1(z,p)\tilde{S}_1(z,p) + \tilde{D}_2(z,p)\tilde{S}_2(z,p). \tag{14}$$

Using the parameterization of hybrid sum-of-squares with positive semidefinite matrices, the modified design problem (9) is transformed, using (14) and similar relations for the other constraints, into an SDP problem. Again, we are forced to set the degrees of the sum-of-squares from (14) (which can theoretically be unbounded); however, practically optimal solutions are obtained with small degrees. Denoting $\tilde{K} = 2\lfloor K/2 + 1 \rfloor$, we have used the degrees $(N, \tilde{K})$ for $S_0(z,p)$, $(N-1, \tilde{K}-2)$ for $S_1(z,p)$ and $(N, \tilde{K}-2)$ for $S_2(z,p)$. (Note that the maximum degree of the variable p of a sum-of-squares must be even.)

Example 2. We take $\tilde{p}_l = 0.019$, $\tilde{p}_u = 0.518$, $\tilde{\Delta} = 0.29$. These choices make the passband edge ω_b vary between $\text{acos}(\tilde{p}_u + \tilde{\Delta}) \approx 0.2\pi$ and $\text{acos}(\tilde{p}_l + \tilde{\Delta}) \approx 0.4\pi$, the same values as in Example 1. However, since the transition band has variable width (see again Figure 3), the stopband edge varies between $\text{acos}(\tilde{p}_u - \tilde{\Delta}) = 0.4268\pi$ and $\text{acos}(\tilde{p}_l - \tilde{\Delta}) = 0.5874\pi$ (in Example 1 it varied between 0.4π and 0.6π). The average width of the transition band is 0.199π, i.e. virtually the same value as in Example 1 (in Figure 3, the areas inside the solid and dashed lines are practically equal). Taking $\gamma_p = 0.01$, we find again, for each K, the minimum degree N producing $\gamma_s \le 0.00316$ after solving (9). The results are in the lower half of Table 1. We notice the lower complexity of the filters, with respect to the constant transition band width solutions obtained by

	K	N	$(K+1)(N+1)$
Constant	2	56	171
transition	3	17	72
band width	4	13	70
Variable	2	24	75
transition	3	15	64
band width	4	13	70

Table 1. Minimal orders satisfying the example design data.

solving (2). In particular, the lowest complexity is obtained for $K = 3$; the magnitude responses of this family of filters is given in Figure 4; their optimal stopband attenuation is $\gamma_s = 0.00303 = -50.37$ dB. (Note that the ripples slightly higher than -50 dB are inside the transition band.)

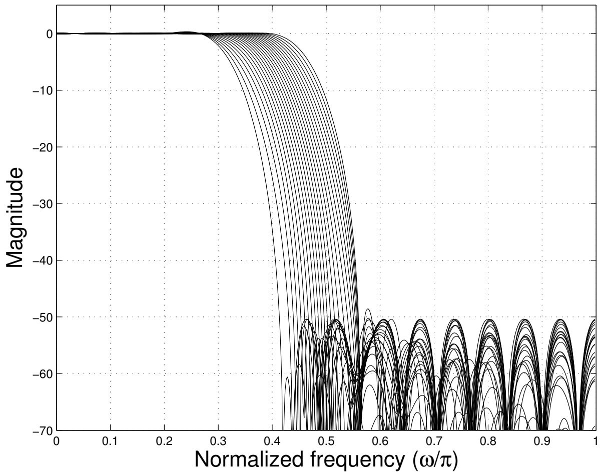

Fig. 4. Frequency responses of adjustable filters designed in Example 2 with $K = 3$, for 25 values of the parameter $p \in [0.019, 0.518]$.

4. CONCLUSIONS

The present paper proposes a 2D polynomial optimization approach for the minimax design of linear-phase adjustable filters. The underlying optimization method relies on the parameterization of positive 2D polynomials via positive semidefinite matrices, leading to sound and efficient SDP algorithms. Unlike previous methods, this approach is not based on discretization. In the standard setup (2), we obtain results that are similar to those given by a linear programming approach [2], but now optimality is guaranteed. We also propose a modification of the standard problem, in which the adjustable filters do not have any longer constant transition width, see (9). Now, the transition band is wider for narrow passband width and narrower for wide passband width. Experimental results show that this choice allows the design of filters with lower complexity, for specifications that are similar to those of the standard problem.

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