A Moulding Technique for the Design of 2-D Orthogonal Filter Banks

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Abstract

We propose a two-stage method for designing nonseparable orthogonal 2-D two-channel filter banks. The first stage consists of solving a convex optimization problem, whose solution is used as initialization for the second stage standard optimization. The resulting filters have good properties in terms of phase linearity in the passband and reconstruction error and compare favorably with previous designs.

EDICS: DSP–TFSR, DSP–FILT

I. Introduction

Nonseparable filters have a distinct place in 2-D signal processing, due to their capability of having passband shapes that are not possible for separable filters. In some applications, this feature compensates their higher implementation complexity. 2-D two-channel filter banks (FB) using quincunx sampling have usually diamond passbands and their design is recognized as a difficult problem, although there are several design techniques in the literature. Most of them are dedicated to biorthogonal FBs, using a lifting scheme implementation [1], [2], [4] or a standard form [3].

We are interested here by orthogonal FBs, for which there are relatively few design methods: [5] employs a lattice parameterization of paraunitary matrices (incomplete in more than one dimension), [6] and [7] are based on direct optimization, initialized with a diamond shaped filter,

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[8] proposes an iterative fitting with a nonfactorable product filter, and [9] uses the Cayley transform allowing a nice characterization of paraunitarity. The first three methods use nonconvex optimization and [9] reduces the problem to solving systems of quadratic equations.

Our aim is to design orthogonal FBs that have approximately linear phase in the passband and possess nearly perfect reconstruction (PR). To this purpose, we propose a two-stage method. The first stage, called moulding and based on semidefinite programming (SDP), has the purpose to "shape" the unique filter defining the FB. The second stage, called polishing, uses standard optimization to enforce the paraunitarity constraints, starting from the typically good initialization given by the first stage.

II. Notations and basic properties

We denote $\boldsymbol{z} = (z_1, z_2)$. For $\boldsymbol{k} = (k_1, k_2) \in \mathbb{Z}^2$, we denote $\boldsymbol{z}^{\boldsymbol{k}} = z_1^{k_1} z_2^{k_2}$. A 2-channel orthogonal filter bank is presented in Fig. 1. The FB is completely determined by the analysis filter on the first channel

$$H(\boldsymbol{z}) = \sum_{\boldsymbol{k}=0}^{\boldsymbol{n}} h_{\boldsymbol{k}} \boldsymbol{z}^{-\boldsymbol{k}} = [1 \; z_1^{-1} \; \dots z_1^{-n_1}] \cdot \boldsymbol{H} \cdot [1 \; z_2^{-1} \; \dots z_2^{-n_2}]^T, \tag{1}$$

where $\boldsymbol{H} \in \mathbb{R}^{(n_1+1)\times(n_2+1)}$ is the coefficients matrix. The second analysis filter is $G(\boldsymbol{z}) = -\boldsymbol{z}^{-\boldsymbol{m}} H(-\boldsymbol{z}^{-1})$, where $\boldsymbol{m} \in \mathbb{Z}^2$ will be defined later. The quincunx sampling is performed according to the matrix

$$\boldsymbol{M} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The polyphase components of $H(\boldsymbol{z})$ result from $H(\boldsymbol{z}) = H_0(\boldsymbol{z}^{\boldsymbol{M}}) + z_1^{-1} H_1(\boldsymbol{z}^{\boldsymbol{M}})$, where $\boldsymbol{z}^{\boldsymbol{M}} = (z_1 z_2, z_1 z_2^{-1})$. The polyphase matrix matrix of the FB is

$$\boldsymbol{U}(\boldsymbol{z}) = \begin{bmatrix} H_0(\boldsymbol{z}) & H_1(\boldsymbol{z}) \\ \boldsymbol{z}^{-\boldsymbol{m}} H_1(\boldsymbol{z}^{-1}) & -\boldsymbol{z}^{-\boldsymbol{m}} H_0(\boldsymbol{z}^{-1}) \end{bmatrix} = \sum_{\boldsymbol{i}=0}^{\boldsymbol{p}} \boldsymbol{U}_{\boldsymbol{i}} \boldsymbol{z}^{-\boldsymbol{i}}, \tag{2}$$

with $\boldsymbol{m}$ chosen such that $\boldsymbol{R}(\boldsymbol{z})$ is causal; its degree is $\boldsymbol{p}$ (if $n_1 = n_2$ and $\boldsymbol{H}$ is a full matrix, then $\boldsymbol{p} = \boldsymbol{n}$) and the matrix coefficients $\boldsymbol{U}_{\boldsymbol{i}}$ have size 2×2. In an orthogonal FB, the polyphase matrix is paraunitary, i.e.

$$\boldsymbol{U}(\boldsymbol{z}) \boldsymbol{U}^T(\boldsymbol{z}^{-1}) = \boldsymbol{I}, \tag{3}$$

a property which ensures perfect reconstruction (PR) of the FB (the output signal is a delayed version of the input one).

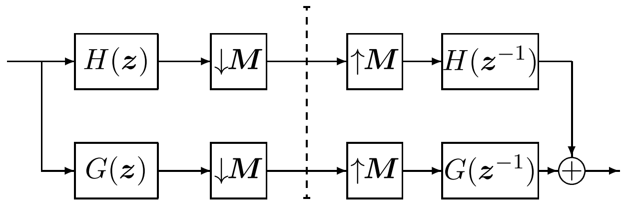

Fig. 1. 2-channel orthogonal filter bank.

Typically, FBs filters are required to satisfy some regularity constraints. For brevity, we write $H(\boldsymbol{\omega})$ for $H(e^{j\boldsymbol{\omega}})$. We say that $H(\boldsymbol{z})$ has order L regularity if

$$\left.\frac{\partial^{\ell_1+\ell_2} H(\boldsymbol{\omega})}{\partial \omega_1^{\ell_1} \partial \omega_2^{\ell_2}}\right|_{\omega_1=\omega_2=\pi} = 0, \quad \forall \ell_1, \ell_2 \geq 0, \; \ell_1 + \ell_2 \leq L-1.$$

This condition is equivalent to

$$\sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} (-1)^{k_1+k_2} k_1^{\ell_1} k_2^{\ell_2} h_{k_1,k_2} = 0. \tag{4}$$

III. Optimization principle

The main obstacle against optimizing the filter $H(\boldsymbol{z})$, for example by minimizing its energy in the stopband, is the nonconvexity of the orthogonality constraint (3), which is quadratic in the coefficients of the filter. In 1-D, the obstacle can be overcome by working with the squared magnitude $P(\boldsymbol{\omega}) = |H(\boldsymbol{\omega})|^2$, in which the problem is convex, and then retrieving $H(\boldsymbol{z})$ by spectral factorization. This is impossible in 2-D, due to the lack of a spectral factorization property for nonnegative polynomials.

Although we cannot eliminate the obstacle, we propose a two stages approach which partly avoids it. The first stage consists of solving a convex optimization problem, which should produce a filter that nearly satisfies (3). The second is a nonconvex optimization problem in which (3) is explicitly imposed, solved via direct optimization and initialized with the result of the first stage.

We discuss here the first stage, as containing our main contribution. We relax the paraunitarity condition (3) to the convex constraint

$$\|\boldsymbol{U}(\boldsymbol{z})\|_\infty \leq 1. \tag{5}$$

We remind that $\|\boldsymbol{U}(\boldsymbol{z})\|_\infty$ is the maximum value of a singular value of $\boldsymbol{U}(\boldsymbol{\omega})$, for $\boldsymbol{\omega} \in [-\pi, \pi]^2$. The condition (5) can be imposed as a linear matrix inequality (LMI), as shown in the next section. However, minimizing the stopband energy subject to the constraints (5) and (4) leads to the trivial result $H(\boldsymbol{z}) = 0$. Imagining the constraint (5) as a mould that should shape the "exterior" of the filter, we have to pump as much energy into the filter such as the mould is filled, case in which *both* singular values of $\boldsymbol{U}(\boldsymbol{\omega})$ would be equal to 1 at *all* frequencies. (Due to the form (2), for any fixed $\boldsymbol{\omega}$, the two singular values of $\boldsymbol{U}(\boldsymbol{\omega})$ are equal.) In the framework of convexity, this can be attempted by minimizing the passband error of $H(\boldsymbol{z})$ with respect to an ideal linear phase response. (Note that maximizing the passband energy is not convex.) Since in a convex optimization problem the optimum is typically on the boundary of the feasible domain, there are reasons to expect that, in this context, (5) is a good approximation of (3). Let

$$\mathcal{D} = \{\boldsymbol{\omega} \in [-\pi, \pi]^2 \mid |\omega_1| + |\omega_2| \leq \omega_p\} \tag{6}$$

be the diamond passband, with some given $\omega_p < \pi$. Let the ideal linear phase response be $D(\boldsymbol{\omega}) = \sqrt{2}e^{-j(\tau_1\omega_1+\tau_2\omega_2)}$, where τ_1, τ_2 are given group delays. The passband error is

$$E_p = \int_{\mathcal{D}} |H(\boldsymbol{\omega}) - D(\boldsymbol{\omega})|^2 d\boldsymbol{\omega}. \tag{7}$$

So, the first (moulding) stage consists of minimizing (7) subject to the constraints (5) and (4).

IV. Algorithmic details

We give here some details on the optimization process. Let $N = (n_1 + 1)(n_2 + 1)$ and $\boldsymbol{h} \in \mathbb{R}^N$ be the vector of filter coefficients obtained by vectorizing (stacking the columns of) the matrix $\boldsymbol{H}$ from (1). The passband error (7) has the form

$$E_p = \boldsymbol{h}^T \boldsymbol{\Gamma} \boldsymbol{h} - 2\boldsymbol{\varphi}^T \boldsymbol{h} + c, \tag{8}$$

with $\boldsymbol{\Gamma} \in \mathbb{R}^{N\times N}$, $\boldsymbol{\Gamma} \succ 0$, $\boldsymbol{\varphi} \in \mathbb{R}^N$ and $c = 2\omega_b^2$. Since the constant elements of (8) result by simple computations, we give directly their expressions. Denote

$$\beta(a_1, a_2) = 8 \cdot \frac{\sin\frac{(a_1+a_2)\omega_b}{2}}{a_1 + a_2} \cdot \frac{\sin\frac{(a_1-a_2)\omega_b}{2}}{a_1 - a_2}.$$

Then, with $\kappa_1, \mu_1, \nu_1 \in 0 : n_1$, $\kappa_2, \mu_2, \nu_2 \in 0 : n_2$, we have

$$\begin{aligned} \boldsymbol{\Gamma}_{(n_1+1)\mu_2+\mu_1,(n_1+1)\nu_2+\nu_1} &= \beta(\nu_1 - \mu_1, \nu_2 - \mu_2), \\ \boldsymbol{\varphi}_{(n_1+1)\kappa_1+\kappa_2} &= \beta(\kappa_1 - \tau_1, \kappa_2 - \tau_2). \end{aligned}$$

The constraint (5) can be implemented using an SDP form of the Bounded Real Lemma (BRL) for trigonometric polynomials with matrix coefficients. We present this result without proof, since it is similar to that from the scalar coefficient case [10, Th.3]; see also [11] for a more general BRL. Let $\widehat{\boldsymbol{U}} \in \mathbb{R}^{2P\times 2}$, with $P = (p_1+1)(p_2+1)$, be the matrix obtained by stacking the matrix coefficients $\boldsymbol{U_i}$ from (2) in column major order, i.e.

$$\widehat{\boldsymbol{U}} = [\boldsymbol{U}_{0,0}^T \ \ldots \ \boldsymbol{U}_{p_1,0}^T \ \boldsymbol{U}_{0,1}^T \ \ldots \boldsymbol{U}_{p_1,p_2}^T]^T.$$

Theorem 1: The inequality $\boldsymbol{U}(\boldsymbol{\omega}) < 1$ holds for all $\boldsymbol{\omega} \in [-\pi, \pi]^2$ if there exists a matrix $\boldsymbol{Q} \in 2P \times 2P$ such that

$$\begin{bmatrix} \boldsymbol{Q} & \widehat{\boldsymbol{U}} \\ \widehat{\boldsymbol{U}}^T & \boldsymbol{I}_2 \end{bmatrix} \succeq 0, \tag{9}$$

and

$$\mathrm{TR}[(\boldsymbol{\Theta}_{i_2} \otimes \boldsymbol{\Theta}_{i_1} \otimes \boldsymbol{I}_2) \cdot \boldsymbol{Q}] = \delta_{i_1 i_2} \boldsymbol{I}_2, \quad \forall i_1 \in 0 : p_1, \ i_2 \in 0 : p_2, \tag{10}$$

where $\boldsymbol{\Theta}_{i_\ell} \in \mathbb{R}^{(p_\ell+1)\times(p_\ell+1)}$, $\ell = 1, 2$, is the elementary Toeplitz matrix with ones on diagonal i_ℓ and zeros elsewhere, $\otimes$ is the Kronecker product, δ is the Kronecker symbol and TR is a trace operator on a 2×2 block level; more precisely, if $\boldsymbol{A} = [\boldsymbol{A}_{s,t} \in \mathbb{R}^{2\times 2}]_{s,t=0:P-1}$, then $\mathrm{TR}[\boldsymbol{A}] = \sum_{s=0}^{P-1} \boldsymbol{A}_{s,s}$. ■

Note that (9) implies $\boldsymbol{Q} \succeq 0$ and that (9)–(10) amount to a linear matrix inequality, since the coefficients of the filter $H(\boldsymbol{z})$ enter linearly in $\widehat{\boldsymbol{U}}$. By reformulating the quadratic criterion as a second order constraint, we conclude that the moulding stage optimization presented at the end of the previous section consists of solving the SDP problem

$$\begin{array}{rl} \min & \alpha \\ \text{subject to} & \|\boldsymbol{\Gamma}^{1/2}\boldsymbol{h} - \boldsymbol{\Gamma}^{-1/2}\boldsymbol{\varphi}\| \leq \alpha \\ & (9), \ (10), \ (4) \end{array} \tag{11}$$

In the second (polishing) stage, we impose explicitly the paraunitarity condition (3), in the typical form involving the squared magnitude of the filter (1). Let

$$R(\boldsymbol{z}) = 1 - H(\boldsymbol{z})H(\boldsymbol{z}^{-1}) = \sum_{\boldsymbol{k}=-\boldsymbol{n}}^{\boldsymbol{n}} r_{\boldsymbol{k}} \boldsymbol{z}^{-\boldsymbol{k}} \tag{12}$$

and consider the vector $\boldsymbol{v} = [r_{k_1,k_2}]_{k_1+k_2=\text{even}}$. The condition (3) is equivalent to $\boldsymbol{v} = 0$. For practical implementation of the second stage, we consider a tolerance ε and solve the optimization problem

$$\begin{array}{rl} \min & \alpha \\ \text{s.t.} & \|\boldsymbol{\Gamma}^{1/2}\boldsymbol{h} - \boldsymbol{\Gamma}^{-1/2}\boldsymbol{\varphi}\| \leq \alpha \\ & \|\boldsymbol{v}\| \leq \varepsilon, \ (4) \end{array} \tag{13}$$

using a direct method, initialized with the result of (11).

V. Results

We have implemented the SDP problem (11) using the CVX library [12]. We have used CVX architecture to build a variable type that correspond to a BRL constraint as that from (9)–(10), making the programming more modular. The nonconvex problem (13) has been solved using Matlab function `fmincon`. The programs generating the examples given in this section can be found at `http://www.schur.pub.ro/Idei2007/Software/ofb2d.zip`.

Example 1. We take the same design data as in [8]: $n_1 = n_2 = 5$, $L = 2$. The choice of the group delays τ_1, τ_2 may significantly affect the results. We have tried many combinations of values and the single clear conclusion is that it is better to take $\tau_1 = \tau_2$. Otherwise, our recommendation is to try a few values that the common sense of a filter designer would accept as possibly good and settle for the best. For the current design data, we report the results for $\tau_1 = \tau_2 = 2$, although some values between 1.9 and 2 can give better values of the criterion (7). The choice of the passband edge parameter ω_b seems less sensitive; we have tried values between 0.3π and 0.7π and the resulting filters have relatively small variations; we report the result for $\omega_b = 0.5\pi$. The first (moulding) stage SDP problem (11) produces a filter whose polyphase matrix $\boldsymbol{U}(\boldsymbol{\omega})$ has singular values larger than 0.9 and nearly equal to 1 in the passband; the orthogonality error is $\|\boldsymbol{v}\| = 0.058$. For a tolerance $\varepsilon = 10^{-5}$, the second (polishing) stage (13) produces the filter from Table II; its frequency response is shown in Fig. 2 and the passband group delays for the two variables are given in Fig. 3. Table I presents some performance indicators. By e_1 and e_2 we denote the average group delay error in the passband. Setting $\varepsilon = 10^{-6}$, the function `fmincon` reaches the maximum number of criterion evaluations (set to 100000) without satisfying the orthogonality constraint; however, the value $\|\boldsymbol{v}\|$ is smaller than in the first design, see the second line of Table I. (These orthogonality error values are typical for the designs we have tried.) The third line of the table corresponds to the design from [8]; the passband error energy E_p and the group delay errors e_1 and e_2 have been computed using as τ_1 and τ_2 the average passband group delays. It is clear that our designs have better phase linearity and better orthogonality error.

Example 2. We aim now a comparison with the designs from [9]. There, the variable is the polyphase matrix (2), whose coefficients $\boldsymbol{U_i}$ are all full matrices. The resulting filter $H(\boldsymbol{z})$ has diamond-like support. It is relatively easy to modify the problems (11) and (13) such that the

TABLE I

FILTER PERFORMANCE FOR EXAMPLE 1.

Method	τ_1	τ_2	E_p	$\|v\|$	e_1	e_2
Proposed, $\varepsilon = 10^{-5}$	2	2	0.0011	$1.0 \cdot 10^{-5}$	0.0091	0.0106
Proposed, $\varepsilon = 10^{-6}$	2	2	0.0022	$3.1 \cdot 10^{-6}$	0.0160	0.0165
[8]	3.174	1.915	0.0051	$8.0 \cdot 10^{-4}$	0.0508	0.0222

TABLE II

FILTER COEFFICIENTS, EXAMPLE 1, $\varepsilon = 10^{-5}$.

$n_1 \backslash n_2$	0	1	2	3	4	5
0	−0.00067619	−0.01403764	−0.00232043	−0.01744734	0.00255490	0.00021092
1	−0.01320679	−0.03454530	0.18413494	−0.04341675	0.00373691	−0.00049540
2	−0.00005543	0.18396150	0.90148550	0.22341469	−0.03416172	0.00063644
3	−0.01072183	−0.04946891	0.23814007	−0.04149407	−0.03810096	0.01063607
4	0.00136953	−0.01186902	−0.03385030	−0.02054726	0.01910765	−0.00080648
5	0.00076616	0.00568725	−0.00083089	0.00669701	−0.00031645	0.00006360

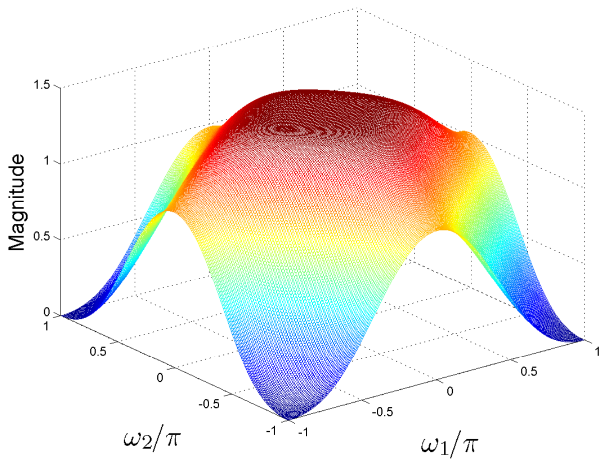

Fig. 2. Frequency response of filter designed in Example 1, $\varepsilon = 10^{-5}$.

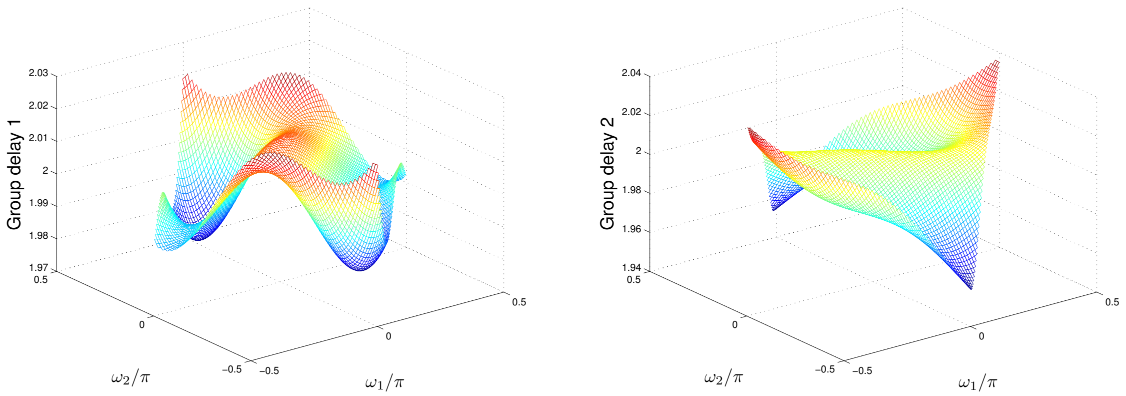

Fig. 3. Group delays in passband for the filter designed in Example 1, $\varepsilon = 10^{-5}$.

TABLE III

FILTER PERFORMANCE FOR EXAMPLE 2.

Method	τ_1	τ_2	E_p	$\|v\|$	e_1	e_2
Proposed, $\varepsilon = 10^{-5}$	2.2	2	0.0020	$1.0 \cdot 10^{-5}$	0.0223	0.0179
Proposed, $\varepsilon = 10^{-6}$	2.2	2	0.0025	$3.5 \cdot 10^{-6}$	0.0320	0.0216
[9, Table II], $L = 3$	1.765	2.887	0.0119	$2.0 \cdot 10^{-6}$	0.0686	0.0584

variable is $\boldsymbol{U}(\boldsymbol{z})$, which is more efficient than forcing $H(\boldsymbol{z})$ to have zero coefficients in the desired positions, due to the lower degree of $\boldsymbol{U}(\boldsymbol{z})$. As in Example 4 from [9], we have taken $p_1 = 3$, $p_2 = 2$, which leads to $n_1 = 6$, $n_2 = 5$. With $L = 2$, $\tau_1 = 2.2$, $\tau_2 = 2$, we have obtained the results in Tables III (performance indicators), IV (filter coefficients) and Fig. 4 (frequency response), 5 (group delays). The same descriptions as for Example 1 apply. The singular values of $\boldsymbol{U}(\boldsymbol{\omega})$ after the moulding stage are larger than 0.95 and the corresponding orthogonality error is $\|\boldsymbol{v}\| = 0.039$. Comparing with the result from [9], we see that our design has better phase linearity, but slightly worse orthogonality error. The design from [9] has also the advantage of maximum regularity $L = 3$. However, our method is more flexible in the choice of the support of the filter and does not rely entirely on solving systems of quadratic equations, whose difficulty increases sharply with the filter order.

VI. Concluding remarks

We have proposed a two-stage method for the design of orthogonal 2-D filter banks with almost linear phase in the passband. The distinguishing feature of the method is the relaxation of the paraunitarity constraint to a bounded norm relation that can be expressed as a linear matrix

TABLE IV

FILTER COEFFICIENTS, EXAMPLE 2, $\varepsilon = 10^{-6}$.

$n_1 \backslash n_2$	0	1	2	3	4	5
0	0	0	−0.01337619	0	0	0
1	0	−0.03212923	0.05490459	−0.07311778	0	0
2	0.00087270	0.13416487	0.86051842	0.22669929	−0.02593046	0
3	−0.00558717	−0.04071064	0.41134676	−0.00000002	−0.02407943	0.00257986
4	0	−0.01029057	0.00949007	−0.08915925	0.01377610	0.00066609
5	0	0	0.00387552	0.00513376	0.00329594	0
6	0	0	0	0.00126992	0	0

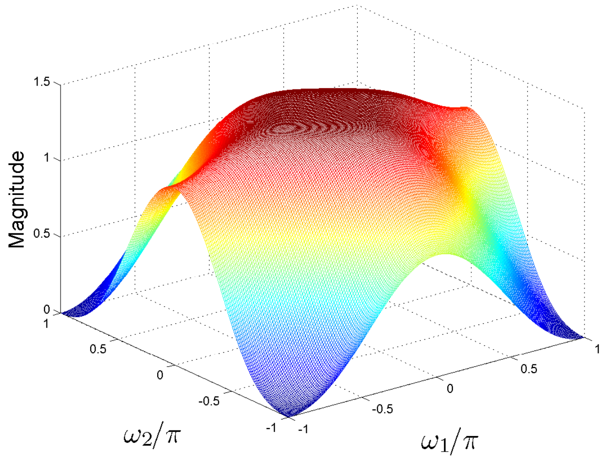

Fig. 4. Frequency response of filter designed in Example 2, $\varepsilon = 10^{-6}$.

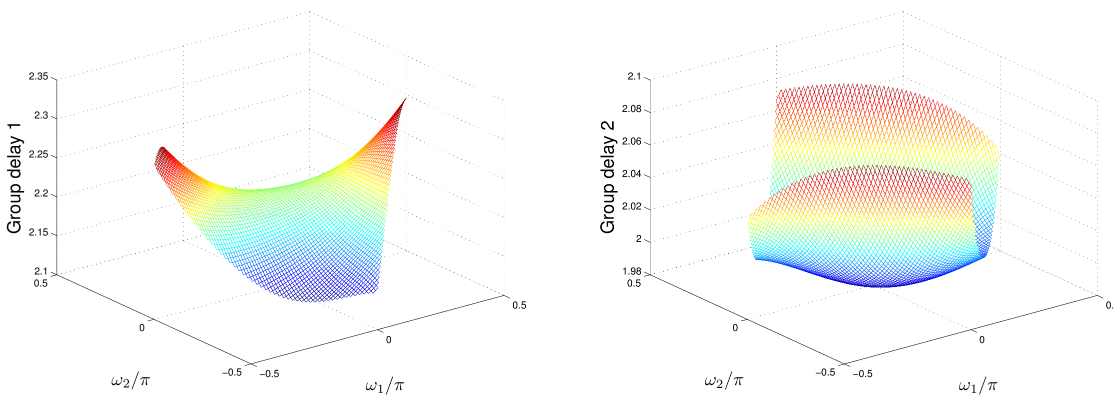

Fig. 5. Group delays in passband for the filter designed in Example 2, $\varepsilon = 10^{-6}$.

inequality. This not only makes the design problem convex, but also gives good approximations of the solution that can be further refined via standard optimization. This technique, called here moulding, may be useful for other related problems, for example in the case of oversampled filter banks that generate tight frames.

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