Positive hybrid real-trigonometric polynomials and applications*

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Abstract

The paper deals with positivity issues of two variable hybrid real-trigonometric polynomials, regarded as a special case of multivariate positive polynomials. Parameterizations and sum-of-squares representations are adapted to the situation where the two variables are one real and the other complex, the latter taking values on the unit circle. Applications in control and signal processing are subsequently described: robust stability of discrete-time systems, design of adjustable FIR filters, absolute stability of time-delay systems.

1 Introduction

Hybrid polynomials depend on both real and complex variables, the latter ones taking values on the unit circle. We deal with the case where the polynomial has real values and are especially interested by positivity conditions. For a simpler discussion, we consider only polynomials with two variables, one real and the other complex. (Generalizations are straightforward.) Such a hybrid polynomial has the form

$$R(t, z) = \sum_{k_1=0}^{n_1} \sum_{k_2=-n_2}^{n_2} r_{k_1,k_2} t^{k_1} z^{-k_2}, \tag{1}$$

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with $t \in \mathbb{R}$, $z \in \mathbb{C}$. The symmetry relation

$$r_{k_1,-k_2} = r^*_{k_1,k_2} \tag{2}$$

implies that the polynomial (1) takes real values on $\mathbb{R} \times \mathbb{T}$, where $\mathbb{T}$ is the unit circle.

A first purpose of our work is to characterize hybrid polynomials that are sum-of-squares, i.e. that can be written as

$$R(t,z) = \sum_{\ell=1}^{\nu} H_\ell(t,z) H^*_\ell(t,z^{-1}), \tag{3}$$

It is evident that in this case the polynomial (1) has an even degree in t; we denote $n_1 = 2m_1$. In (3), each polynomial $H_\ell(t,z)$ is causal in z (and named simply causal), i.e. it has the expression (for clarity, we omit the index ℓ)

$$H(t,z) = \sum_{k_1=0}^{m_1} \sum_{k_2=0}^{n_2} h_{k_1,k_2} t^{k_1} z^{-k_2}. \tag{4}$$

It is important to note from the beginning that not all positive hybrid polynomials are sum-of-squares. This is a consequence of the similar result for multivariate polynomials. (The notable exception is the case of trigonometric polynomials.)

2 Gram matrix parameterization

We denote

$$\boldsymbol{\psi}_n(t) = [1 \ t \ t^2 \ \ldots \ t^n]^T \tag{5}$$

the standard basis for degree n polynomials and

$$\boldsymbol{\psi}_{m_1,n_2}(t,z) = \boldsymbol{\psi}_{n_2}(z) \otimes \boldsymbol{\psi}_{m_1}(t) \tag{6}$$

the basis for bivariate hybrid polynomials. The index (m_1, n_2) will be omitted when clear from the context. We denote $N = (m_1+1)(n_2+1)$ the number of monomials in the basis. A causal hybrid polynomial (4) can be written as

$$H(t,z) = \boldsymbol{\psi}^T(t,z^{-1})\boldsymbol{h}, \tag{7}$$

where $\boldsymbol{h} \in \mathbb{C}^N$ is a vector containing the coefficients of $H(t,z)$ ordered as corresponding to the basis (6).

A Hermitian matrix $\boldsymbol{Q}$ is called a *Gram* matrix associated with the hybrid polynomial (1) if

$$R(t,z) = \boldsymbol{\psi}^T(t,z^{-1}) \cdot \boldsymbol{Q} \cdot \boldsymbol{\psi}(t,z). \tag{8}$$

Theorem 1. *In (8), the relation between the coefficients of the hybrid polynomial $R(t,z)$ and the elements of the matrix $\boldsymbol{Q}$ is*

$$r_{k_1,k_2} = \mathit{trace}[(\boldsymbol{\Theta}_{k_2} \otimes \boldsymbol{\Upsilon}_{k_1}) \cdot \boldsymbol{Q}], \tag{9}$$

where $\boldsymbol{\Theta}_{k_2} \in \mathbb{R}^{(n_2+1)\times(n_2+1)}$ *is the elementary Toeplitz matrix with ones on diagonal* k_2 *and zeros elsewhere, and* $\boldsymbol{\Upsilon}_{k_1} \in \mathbb{R}^{(m_1+1)\times(m_1+1)}$ *is the elementary Hankel matrix with ones on antidiagonal* k_1 *and zeros elsewhere.*

Proof. The relation (8) is equivalent to

$$\begin{aligned} R(t,z) &= \operatorname{trace}[\boldsymbol{\psi}(t,z)\cdot\boldsymbol{\psi}^T(t,z^{-1})\cdot\boldsymbol{Q}] \\ &= \operatorname{trace}\left[\Big(\boldsymbol{\psi}(z)\boldsymbol{\psi}^T(z^{-1})\Big)\otimes\Big(\boldsymbol{\psi}(t)\boldsymbol{\psi}^T(t)\Big)\cdot\boldsymbol{Q}\right] \\ &= \sum_{k_1=0}^{n_1}\sum_{k_2=-n_2}^{n_2}\operatorname{trace}\left[(\boldsymbol{\Theta}_{k_2}\otimes\boldsymbol{\Upsilon}_{k_1})\cdot\boldsymbol{Q}\right]t^{k_1}z^{-k_2} \end{aligned}$$

By identifying this expression with (1), the equality (9) results. ■

The sum-of-squares polynomials are characterized by a positive semidefinite Gram matrix.

Theorem 2. *A hybrid polynomial (1) is sum-of-squares if and only if there exists a positive semidefinite matrix* $\boldsymbol{Q} \in \mathbb{C}^{N\times N}$ *such that (8) holds.*

Proof. The proof is standard and is based on the relation between the eigenvalue decomposition of $\boldsymbol{Q}$ and the sum-of-squares definition (3), via (7), see e.g. [4]. ■

We also note that if the sum-of-squares $R(t,z)$ has real coefficients, then the matrix $\boldsymbol{Q} \succeq 0$ from the above theorem has also real coefficients.

3 Positivity on domains

Sum-of-squares polynomials appear in characterizations of hybrid polynomials that are positive on domains defined by the positivity of some polynomials, i.e.

$$\mathcal{D} = \{(t,z) \in \mathbb{R}\times\mathbb{T} \mid D_\ell(t,z) \geq 0,\ \ell = 1:L\}, \tag{10}$$

where $D_\ell(t,z)$ are hybrid polynomials defined as in (1). We assume that $\mathcal{D}$ is bounded and that $D_L(t,z) = (t-a)(b-t)$, which implies $\mathcal{D} \subset [a,b]\times\mathbb{T}$. (For any bounded $\mathcal{D}$, the polynomial $(t-a)(b-t)$ can be explicitly added to the definition (10), if not already present.)

Theorem 3. *Under the above assumptions, a polynomial (1) is positive on* $\mathcal{D}$*, i.e.* $R(t,z) > 0$, $\forall (t,z) \in \mathcal{D}$*, if and only if there exist sum-of-squares* $S_\ell(t,z)$, $\ell = 0:L$*, such that*

$$R(t,z) = S_0(t,z) + \sum_{\ell=1}^{L} D_\ell(t,z)\cdot S_\ell(t,z). \tag{11}$$

If the polynomials $R(t,z)$ *and* $D_\ell(t,z)$ *have real coefficients, then the sum-of-squares* $S_\ell(t,z)$ *have also real coefficients.*

Proof. The proof is based on results from [8, 6] and on the transformation described in [3]. ∎

We note that the degrees of the sum-of-squares may be larger than the degree of $R(t, z)$. In the particular case where $\mathcal{D} = [a, b] \times \mathbb{T}$, the relation (11) has the form

$$R(t, z) = S_0(t, z) + (t - a)(b - t)S_1(t, z). \tag{12}$$

4 Applications

We examine here several applications of sum-of-squares hybrid polynomials parameterization and give some examples of use.

4.1 Stability problems

A first example is the stability of a 2-D continuous-discrete-time system whose transfer function has the denominator $A(s, z)$. This denominator must be Hurwitz-Schur, i.e. $A(s, z) \neq 0$, $\forall \mathrm{Re}(s) \geq 0, |z| \leq 1$. The test can be reduced to some 1-D conditions and the 2-D "border" condition $A(jt, z) \neq 0$, $\forall t \in \mathbb{R}, z \in \mathbb{T}$. This condition can be transformed into

$$R(t, z) = A(jt, z)A(-jt, z^{-1}) > 0, \ \forall t \in \mathbb{R}, z \in \mathbb{T}. \tag{13}$$

We cannot test exactly this positivity condition, so we change it into a sufficient condition by requiring that $R(t, z)$ is *a strictly positive sum-of-squares*. Similarly to the 2-D discrete systems case treated in [2], we expect that the sum-of-squares condition is practically necessary.

To test the sharpness of the condition, we can work in (3) with polynomials $H_\ell(t, z)$ whose degree in z is larger than n_2. (The degree in t cannot be made larger.) This can be done in combination with the condition that $(1 + t^2)^\kappa R(t, z)$ is sum-of-squares, where κ is a small positive integer.

A second typical problem is that of robust stability. Let us consider the (1D) discrete-time system whose transfer function has the denominator

$$A(\tau, z) = \sum_{k=0}^{n} p_k(\tau)z^{-k}, \tag{14}$$

where $p_k(\tau)$ are polynomials in the unknown parameter $\tau \in [a, b]$. We want to test if the polynomial is Schur for all admissible values of the parameter, i.e. $A(\tau, z) \neq 0$, $\forall |z| \leq 1$, $\forall \tau \in [a, b]$. Different algorithms for this problem have been proposed in [5, 10]. As above, we can transform this into a positivity problem, by requiring that the hybrid polynomial $R(\tau, z) = A(\tau, z)A(\tau, z^{-1})$ has the form (12) (which is a sufficient condition).

The above problems can be easily generalized to more than two variables, for example in the case where the polynomial coefficients of (14) depend on more than one parameter.

4.2 Design of adjustable FIR filters

Another possible application is that of adjustable FIR filters, which have a transfer function defined by

$$G(b,z) = \sum_{k=0}^{K} (b - b_0)^k H_k(z), \tag{15}$$

where $H_k(z)$, $k = 0 : K$, are FIR filters, $b_0 \in \mathbb{R}$ is a constant and $b \in \mathbb{R}$ is variable. If the filters $H_k(z)$ have linear phase, the same length, and are all of the same type (symmetric or antisymmetric), like e.g. in [7], then we can assume that they are actually zero-phase and thus the transfer function (15) is a hybrid polynomial similar to (1), in the variables b and z.

Let us examine the minimax design of an adjustable lowpass filter of fixed order n. The optimization problem can be formulated e.g. as

$$\begin{array}{ll} \min & \gamma_s \\ \text{s.t.} & 1 - \gamma_p \le G(b, e^{j\omega}) \le 1 + \gamma_p, \ \forall \omega \in [0, b - \Delta] \\ & -\gamma_s \le G(b, e^{j\omega}) \le \gamma_s, \ \forall \omega \in [b + \Delta, \pi] \end{array} \tag{16}$$

Here, γ_b is a prescribed passband error bound and the stopband error γ_s is minimized. A feasibility problem could be solved as well, with fixed γ_s. The parameter b takes values in an interval $[b_l, b_u]$ and 2Δ is the width of the transition band. The parameter b_0 can have any fixed value, e.g. $b_0 = (b_l + b_u)/2$. The conditions $b_l - \Delta > 0$ and $b_u + \Delta < \pi$ must be satisfied.

We can formulate a similar problem, on the filter

$$G(t,z) = \sum_{k=0}^{K} (t - t_0)^k H_k(z), \tag{17}$$

with $t \in [-t_l, t_u]$ (understood as $t = \cos\theta$). The problem is

$$\begin{array}{ll} \min & \gamma_s \\ \text{s.t.} & 1 - \gamma_p \le G(t, e^{j\omega}) \le 1 + \gamma_p, \ \forall \cos\omega \in [t + \Delta, 1] \\ & -\gamma_s \le G(t, e^{j\omega}) \le \gamma_s, \ \forall \cos\omega \in [-1, t - \Delta] \end{array} \tag{18}$$

Now, the conditions $t_u + \Delta < 1$ and $t_l - \Delta > -1$ must be satisfied. The parameter t_0 can have any fixed value, e.g. $t_0 = (t_l + t_u)/2$. The width of the transition band is $\text{acos}(t + \Delta) - \text{acos}(t - \Delta)$ and thus it is dependent on the value of the parameter t. The problem (18) has the appealing feature that the transition band is larger when the passband or the stopband are narrow (and a good stopband attenuation is more difficult to obtain).

Each of the four inequalities from (18) can be expressed via the positivity of a hybrid polynomial (replacing $\cos\omega = (z + z^{-1})/2$) on domains defined by the positivity of hybrid polynomials. The problem bears some resemblance with the approach from [3], dedicated to the design of 2-D FIR filters.

Example. The SDP version of the problem (18) has been implemented using SeDuMi [11]. We give here only one example of design. We set the degree of the

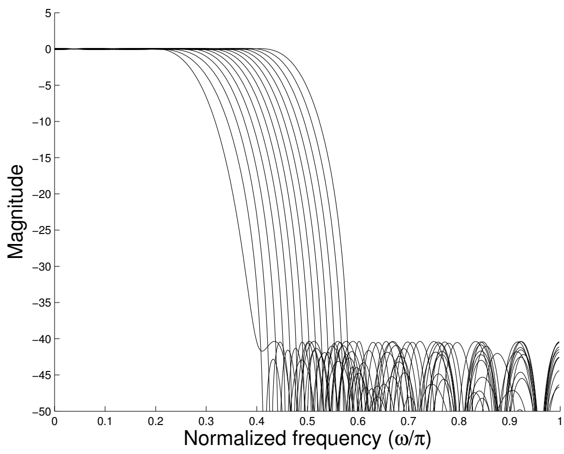

Figure 1. *Frequency responses of a family of adjustable filters.*

filters $H_k(z)$ to be equal to 26 (which means that we have $n_2 = 13$ in (2)) and take $K = 4$. The other design parameters are $t_l = 0$, $t_u = 0.56$, $t_0 = (t_l + t_u)/2$, $\Delta = 0.25$ and $\gamma_p = 0.01$. In the parameterizations (11), we take the degrees of the sum-of-squares such that the degree in t is 6 (i.e greater than the degree of the filter $G(t, z)$, which is 4). With these values, we obtain $\gamma_s = 9.60 \cdot 10^{-3}$, i.e. an attenuation greater than 40 dB in the stopband. The frequency responses of a family of filters $G(t, z)$, with $t = 0 : 0.04 : 0.56$, are shown in figure 1.

4.3 Absolute stability of systems with delays

Positive hybrid real-trigonometric polynomials are naturally showing up in frequency-domain conditions involving time-delay systems. We illustrate this by considering Popov's absolute stability criterion for the following (scalar) control system (see [9] and Problem 6.6 in [1]):

$$\begin{aligned} \dot{x}(t) &= -ax(t) + \phi\left(y(t)\right) \\ y(t) &= x(t) + cx(t-\tau) \end{aligned} \tag{19}$$

where $a > 0$, $c \in \mathbb{R}$, $\tau > 0$ and ϕ is a sector type nonlinearity, $0 \leq \dfrac{\phi(\sigma)}{\sigma} \leq k \leq \infty$.

According to the Popov criterion, since $a > 0$ (stable linear part), the system (19) is asymptotically stable for every nonlinearity satisfying the above inequality if there exists $q \geq 0$ such that Popov's *frequency domain condition* is verified:

$$\frac{1}{k} + \Re\left[(1 + j\omega q)\, G(j\omega)\right] > 0, \quad \forall \omega \in \overline{\mathbb{R}}. \tag{20}$$

Here $G(s)$ is the transfer function of the linear part and it is given by

$$G(s) = \frac{1 + ce^{-s\tau}}{s + a}. \tag{21}$$

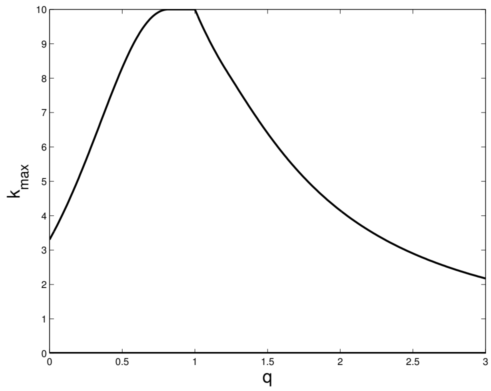

Figure 2. *Maximal values of absolute stability sector.*

The condition that (20) holds for *all* $\tau > 0$, i.e. the absolute stability is delay independent, is equivalent to the positivity of a hybrid polynomial. Indeed, after elementary algebraic manipulations the frequency condition (20) rewrites as

$$\begin{aligned} &2(a + j\omega)(a - j\omega) + k(1 + j\omega q)(a - j\omega)(1 + ce^{-j\omega\tau}) \\ &+ k(1 - j\omega q)(a + j\omega)(1 + ce^{j\omega\tau}) > 0, \quad \forall \omega \in \overline{\mathbb{R}}. \end{aligned}$$

Putting $z = e^{j\omega\tau}$ (since τ can have any value, z and ω are decoupled), this is equivalent to

$$R(\omega, z) = 2(a^2 + \omega^2) + H(\omega, z) + H(-\omega, z^{-1}) > 0, \quad \forall \omega \in \overline{\mathbb{R}}, z \in \mathbb{T}, \tag{22}$$

where $H(\omega, z) = k[a + j(aq - 1)\omega + q\omega^2](1 + cz^{-1})$.

Since each of q and k enter linearly in the coefficients of $R(\omega, z)$, one can easily search for a "best suitable" q (with fixed k) or for the maximal sector of absolute stability k_{max} (with fixed q).

Example. It can be proved [1, Problem 6.6] that the condition (20) holds for any k if $|c| < 1$. However, for $|c| > 1$, the maximal sector of absolute stability k_{max} has a finite value. This value can be estimated by taking values of q and, for each of them, computing the maximal value k for which (22) holds. For example, with $c = 1.1$, we obtain the curve from figure 2.

5 Conclusion

We have presented basic properties of hybrid real-trigonometric polynomials and their applications in filter design and delay-independent absolute stability. Further work will be devoted to enlarge the area of applications and to solve problems with higher complexity.

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