Computing the Controllability Radius: a Semidefinite Programming Approach

B. Dumitrescu, B.C. Şicleru, R. Ştefan

The authors are with Department of Automatic Control and Computers, "Politehnica" University of Bucharest, Spl. Independenţei 313, 060042 Bucharest, Romania, e-mail: bogdan@schur.pub.ro. Work supported by Romanian CNCSIS grant IDEI 309/2007. B.Dumitrescu is also with Tampere International Center for Signal Processing, Tampere University of Technology, Finland.

Abstract

We propose a semidefinite programming (SDP) approach to compute the controllability radius. We transform the initial nonconvex optimization problem into the minimization of the smallest eigenvalue of a bivariate real or trigonometric polynomial with matrix coefficients. A sum-of-squares relaxation leads to the SDP formulation. A similar technique is used for the computation of the stabilizability radius. We extend the approach to the computation of the worst-case controllability radius for systems that depend polynomially on a small number of parameters. Experimental results show that our methods compete well with previous ones in a complexity/accuracy tradeoff.

Keywords: controllability, stabilizability, sum-of-squares polynomials, semidefinite programming.

I. Introduction

We consider a continuous-time system described by the state-space equation $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$, $t \in \mathbb{R}$, where $\boldsymbol{x}$ is the state vector and $\boldsymbol{u}$ is the input vector, with constant matrices $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, $\boldsymbol{B} \in \mathbb{C}^{n\times m}$. Such a system is controllable if

$$\text{rank}[\boldsymbol{A} - \lambda \boldsymbol{I} \ \boldsymbol{B}] = n, \quad \forall \lambda \in \mathbb{C}. \tag{1}$$

Note that controllability is defined identically for a discrete-time system $\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k)$, $k \in \mathbb{Z}$. Since controllability is a generic property, i.e. an arbitrarily small perturbation of the pair $(\boldsymbol{A}, \boldsymbol{B})$ may make an uncontrollable system controllable, one is interested in the robustness of the controllability property. The complex controllability radius is defined [1] as the smallest 2-norm of a perturbation of $(\boldsymbol{A}, \boldsymbol{B})$ that makes the system uncontrollable, i.e.

$$\begin{aligned} \rho(\boldsymbol{A}, \boldsymbol{B}) = \min_{\boldsymbol{\Delta}_A, \boldsymbol{\Delta}_B} \quad & \|[\boldsymbol{\Delta}_A \ \boldsymbol{\Delta}_B]\| \\ \text{s.t.} \quad & (\boldsymbol{A} + \boldsymbol{\Delta}_A, \boldsymbol{B} + \boldsymbol{\Delta}_B) \text{ is uncontrollable} \end{aligned} \tag{2}$$

It was proved [2] that

$$\rho(\boldsymbol{A}, \boldsymbol{B}) = \min_{\lambda \in \mathbb{C}} \sigma_{min}([\boldsymbol{A} - \lambda \boldsymbol{I} \ \boldsymbol{B}]), \tag{3}$$

where the function $\sigma_{min}()$ returns the smallest singular value of its matrix argument. The advantage of (3) is that one minimizes over a single complex variable (λ), while in (2) the variables are matrices ($\boldsymbol{\Delta}_A$, $\boldsymbol{\Delta}_B$).

The first methods for computing the stability radius were based directly on solving (3). Local optimization methods like that from [3] are fast, but may be trapped in a local minimum of the non-convex function $\sigma_{min}([\boldsymbol{A} - \lambda \boldsymbol{I} \ \boldsymbol{B}])$. Methods based on global optimization, e.g. [4], are safer, but their complexity is not guaranteed. The first polynomial-time method (known as Gu's algorithm), based on bisection, was presented in [5]. In a trisection version [6], it was shown to have good accuracy. Its complexity is $O(n^6)$. Recently, the core of this algorithm was made faster in [7], giving a method that works in $O(n^4)$ in average and $O(n^5)$ in the worst case.

It was remarked in [6] that "it does not seem to be known whether Gu's test could be replaced by an LMI-based test". Our contribution is precisely of this nature, transforming (3) into a convex optimization problem. In Section II we show how to transform (3) into the minimization of the smallest eigenvalue of a positive matrix polynomial; there are two implementation approaches, one based on real polynomials, the other on trigonometric polynomials. We use a sum-of-squares relaxation and the associated Gram matrix parameterization, which leads to semidefinite programming (SDP) problems. Both approaches provide SDP problems of complexity at most $O(n^6)$. In Section III, we deal with the related problem of stabilizability radius and show how it can be solved in the same manner, using results on matrix polynomials that are positive on semialgebraic sets. Unlike any previous method, our approach can be extended to the case where $\boldsymbol{A}$ and/or $\boldsymbol{B}$ depend (linearly or even polynomially) on a small number of bounded parameters. Section IV is dedicated to the computation of the smallest controllability radius over the set of parameters. Finally, in SectionV, we compare our method with those from [5] and [7] and show it to compete well in a complexity/accuracy tradeoff. We also present a numerical example illustrating all our contributions.

II. Transformation to sum-of-squares problems

We can transform (3) into a convex optimization problem by "squaring", i.e. by using the equality

$$\sigma_{min}(\boldsymbol{M}) = \sqrt{\lambda_{min}(\boldsymbol{M}\boldsymbol{M}^H)}, \tag{4}$$

where $\boldsymbol{M} \in \mathbb{C}^{n\times(n+m)}$ may be arbitrary and $\lambda_{min}(\boldsymbol{M}\boldsymbol{M}^H)$ is the smallest eigenvalue of $\boldsymbol{M}\boldsymbol{M}^H$; by $\boldsymbol{M}^H$ we denote the complex conjugated transposed of the matrix $\boldsymbol{M}$. In our case, we have $\boldsymbol{M} = [\boldsymbol{A} - \lambda\boldsymbol{I}\ \boldsymbol{B}]$ and so (3) becomes equivalent to the computation of the square root of the smallest eigenvalue of

$$\boldsymbol{P}(\lambda) = |\lambda|^2\boldsymbol{I} - \lambda\boldsymbol{A}^H - \overline{\lambda}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{A}^H + \boldsymbol{B}\boldsymbol{B}^H, \tag{5}$$

for $\lambda \in \mathbb{C}$, a problem that can be equivalently written as

$$\begin{aligned} \tau_o = \max_{\tau \geq 0} \quad & \tau \\ \text{s.t.} \quad & \boldsymbol{P}(\lambda) - \tau \boldsymbol{I} \succeq 0, \quad \forall \lambda \in \mathbb{C} \end{aligned} \tag{6}$$

The controllability radius is $\rho(\boldsymbol{A}, \boldsymbol{B}) = \sqrt{\tau_o}$. In (6) we actually minimize the smallest eigenvalue of $\boldsymbol{P}(\lambda)$.

There are at least two ways to approximate (6) with a problem involving positive polynomials. The first is to use the real and imaginary parts of λ as variables. By putting $\lambda = x + jy$, the function (5) becomes the bivariate matrix polynomial

$$\boldsymbol{P}(x, y) = (x^2 + y^2)\boldsymbol{I} - x(\boldsymbol{A} + \boldsymbol{A}^H) - yj(\boldsymbol{A}^H - \boldsymbol{A}) + \boldsymbol{A}\boldsymbol{A}^H + \boldsymbol{B}\boldsymbol{B}^H. \tag{7}$$

Using $\boldsymbol{P}(x, y)$ in (6) produces the constraint $\boldsymbol{P}(x, y) - \tau \boldsymbol{I} \succeq 0$, which is convex, but hard to implement exactly. However, we can approximate it with a sum-of-squares condition and obtain the problem

$$\begin{aligned} \tau_1 = \max_{\tau \geq 0} \quad & \tau \\ \text{s.t.} \quad & \boldsymbol{P}(x, y) - \tau \boldsymbol{I} \text{ is sum-of-squares} \end{aligned} \tag{8}$$

(This approximation is often named sum-of-squares relaxation and was initiated, for scalar sum-of-squares, in [8].) We remind that a (bivariate) polynomial is sum-of-squares if it can be written as $\sum_{k=1}^{\nu} \boldsymbol{F}_k(x, y)\boldsymbol{F}_k(x, y)^H$ and that not all positive polynomials are sum-of-squares. Since the total degree of $\boldsymbol{P}(x, y)$ is two, the polynomials $\boldsymbol{F}_k(x, y)$ have degree one. Optimization with sum-of-squares polynomials can be transformed into SDP by using the Gram matrix parameterization [9],[10]. The matrix polynomial $\boldsymbol{P}(x, y) - \tau \boldsymbol{I}$ is sum-of-squares if and only if there exists a positive semidefinite matrix $\boldsymbol{Q} \in \mathbb{C}^{3n \times 3n}$ such that

$$\boldsymbol{P}(x, y) - \tau \boldsymbol{I} = \boldsymbol{\Psi}^T(x, y) \cdot \boldsymbol{Q} \cdot \boldsymbol{\Psi}(x, y), \tag{9}$$

with $\boldsymbol{\Psi}(x, y) = [\boldsymbol{I} \ x\boldsymbol{I} \ y\boldsymbol{I}]^T$. Substituting (7) in (9), it results with simple calculations that the matrix $\boldsymbol{Q}$ has the form

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{A}\boldsymbol{A}^H + \boldsymbol{B}\boldsymbol{B}^H - \tau \boldsymbol{I} & -\boldsymbol{A}^H - \boldsymbol{X} & -j\boldsymbol{A}^H - \boldsymbol{Y} \\ -\boldsymbol{A} + \boldsymbol{X} & \boldsymbol{I} & -\boldsymbol{Z} \\ j\boldsymbol{A} + \boldsymbol{Y} & \boldsymbol{Z} & \boldsymbol{I} \end{bmatrix} \tag{10}$$

where the matrices $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathbb{C}^{n\times n}$ are skew-Hermitian ($\boldsymbol{X}^H = -\boldsymbol{X}$, etc.). The problem (8) is equivalent to

$$\begin{aligned} \tau_1 = \ & \max_{\tau \geq 0} \quad \tau \\ & \text{s.t.} \quad (10),\ \boldsymbol{Q} \succeq 0 \end{aligned} \tag{11}$$

which is clearly an SDP problem (the variables are the elements of $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$).

The second transformation of (6) is based on trigonometric polynomials. Let us first remind that the value λ that gives the controllability radius in (3) must belong to the field of values of $\boldsymbol{A}$ [3], i.e. there is a vector $\boldsymbol{v} \in \mathbb{C}^n$, with $\|\boldsymbol{v}\| = 1$, such that $\lambda = \boldsymbol{v}^H \boldsymbol{A}\boldsymbol{v}$. It follows that

$$\begin{aligned} \lambda_{min}(\tfrac{\boldsymbol{A}+\boldsymbol{A}^H}{2}) \leq x \leq \lambda_{max}(\tfrac{\boldsymbol{A}+\boldsymbol{A}^H}{2}), \\ \lambda_{min}(\tfrac{\boldsymbol{A}-\boldsymbol{A}^H}{2j}) \leq y \leq \lambda_{max}(\tfrac{\boldsymbol{A}-\boldsymbol{A}^H}{2j}). \end{aligned} \tag{12}$$

We now write $\lambda = re^{j\omega}$, with $r \geq 0$, $\omega \in \mathbb{R}$. Noting that $r^2 = |\lambda|^2 = x^2 + y^2$, we take advantage of (12) to find nonnegative constants r_1 and r_2 such that $r_1 \leq r \leq r_2$. (Although we often obtain $r_1 = 0$, this inequality will be helpful in other situations.) We introduce the variables $z_1, z_2 \in \mathbb{C}$, with $|z_1| = |z_2| = 1$, such that

$$z_1 = e^{j\omega}, \quad r = r_1 + (r_2 - r_1)[1/2 + (z_2 + z_2^{-1})/4]. \tag{13}$$

(Note that $1/2 + (z_2 + z_2^{-1})/4$ takes values in $[0, 1]$.) In these variables, the function (5) becomes the matrix trigonometric polynomial

$$\boldsymbol{P}(z_1, z_2) = \sum_{k_1=-1}^{1} \sum_{k_2=-2}^{2} \boldsymbol{P}_{k_1,k_2} z_1^{-k_1} z_2^{-k_2}, \tag{14}$$

whose coefficients are symmetric ($\boldsymbol{P}_{-k_1,-k_2} = \boldsymbol{P}_{k_1,k_2}^H$) and are given by

$$\begin{aligned} \boldsymbol{P}_{0,0} &= \left(\tfrac{3}{8}r_1^2 + \tfrac{3}{8}r_2^2 + \tfrac{1}{4}r_1 r_2\right)\boldsymbol{I} + \boldsymbol{A}\boldsymbol{A}^H + \boldsymbol{B}\boldsymbol{B}^H, \\ \boldsymbol{P}_{1,0} &= -\tfrac{1}{2}(r_1 + r_2)\boldsymbol{A}, \\ \boldsymbol{P}_{-1,1} &= -\tfrac{1}{4}(r_2 - r_1)\boldsymbol{A}^H, \\ \boldsymbol{P}_{0,1} &= \tfrac{1}{4}(r_2^2 - r_1^2)\boldsymbol{I}, \\ \boldsymbol{P}_{1,1} &= -\tfrac{1}{4}(r_2 - r_1)\boldsymbol{A}, \\ \boldsymbol{P}_{-1,2} &= \boldsymbol{0}, \\ \boldsymbol{P}_{0,2} &= \tfrac{1}{16}(r_2 - r_1)^2\boldsymbol{I}, \\ \boldsymbol{P}_{1,2} &= \boldsymbol{0}. \end{aligned} \tag{15}$$

The constraint of (6) is thus transformed into the condition

$$\boldsymbol{P}(z_1, z_2) - \tau \boldsymbol{I} \succeq 0. \tag{16}$$

Unlike real polynomials, *all* positive trigonometric polynomials are sum-of-squares, see e.g. [11]. Using the Gram matrix parameterization [12], it results that (16) is equivalent to the existence of $\boldsymbol{Q} \succeq 0$ such that

$$\boldsymbol{P}(z_1, z_2) - \tau \boldsymbol{I} = \boldsymbol{\Psi}^T(z_1^{-1}, z_2^{-1}) \cdot \boldsymbol{Q} \cdot \boldsymbol{\Psi}(z_1, z_2). \tag{17}$$

Here, $\boldsymbol{\Psi}(z_1, z_2)$ is a (matrix) basis for trigonometric polynomials, and may contain, in principle, an arbitrarily large number of terms. In applications, we typically use the minimum number of terms, hence we use in (17) the basis

$$\boldsymbol{\Psi}(z_1, z_2) = [\boldsymbol{I} \;\; z_1\boldsymbol{I} \;\; z_2\boldsymbol{I} \;\; z_1z_2\boldsymbol{I} \;\; z_2^2\boldsymbol{I} \;\; z_1z_2^2\boldsymbol{I}]^T \tag{18}$$

and the size of $\boldsymbol{Q}$ is $6n \times 6n$. We have obtained a second equivalent of (6), namely

$$\begin{aligned} \tau_2 = \quad & \max_{\tau \geq 0} \quad \tau \\ & \text{s.t.} \quad (17),\;\; \boldsymbol{Q} \succeq 0 \end{aligned} \tag{19}$$

Again, this is an SDP problem. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are real, then the coefficients (15) of $\boldsymbol{P}(z_1, z_2)$ are real and hence the matrix $\boldsymbol{Q}$ is also real.

Comment 1. Both problems (11) and (19) are approximations of (6). In the first case, the cause is the theoretical fact that not all positive real polynomials are sum-of-squares. In the second case, the cause is the practical necessity to bound the number of terms in the basis of trigonometric polynomials. In both cases we solve more conservative problems, and thus we should obtain $\tau_1 \leq \tau_o$, $\tau_2 \leq \tau_o$.

It is often possible to obtain a certificate of optimality. We discuss the case of real polynomials. Let $\boldsymbol{Q}_o$ be the optimal Gram matrix (10), obtained by solving (11). Typically, this matrix is singular. Let $\boldsymbol{U} \in \mathbb{C}^{3n \times p}$ be a basis for Ker$\boldsymbol{Q}_o$ (typically p is small); we split $\boldsymbol{U}^T = [\boldsymbol{U}_0^T \; \boldsymbol{U}_1^T \; \boldsymbol{U}_2^T]$, with $\boldsymbol{U}_0$, $\boldsymbol{U}_1$, $\boldsymbol{U}_2$ of size $n \times p$. If $\tau_1 = \tau_o$, then the matrix $\boldsymbol{P}(x, y) - \tau_1 \boldsymbol{I}$ is singular for the (unknown) optimal values x, y; let $\boldsymbol{v} \in \mathbb{C}^n$ be an eigenvector corresponding to the zero eigenvalue. It follows from (9) that $\boldsymbol{Q}_o \boldsymbol{\Psi}(x, y)\boldsymbol{v} = 0$, i.e. $\boldsymbol{\Psi}(x, y)\boldsymbol{v} \in$ Ker$\boldsymbol{Q}_o$ and, taking into account that

$\boldsymbol{\Psi}(x,y) = [\boldsymbol{I}\ x\boldsymbol{I}\ y\boldsymbol{I}]^T$, it exists a vector $\boldsymbol{w} \in \mathbb{C}^p$ such that

$$\begin{bmatrix} \boldsymbol{v} \\ x\boldsymbol{v} \\ y\boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \boldsymbol{U}_0 \\ \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \end{bmatrix} \cdot \boldsymbol{w}. \tag{20}$$

Hence, we have $\boldsymbol{v} = \boldsymbol{U}_0\boldsymbol{w}$ and so $x\boldsymbol{U}_0\boldsymbol{w} = \boldsymbol{U}_1\boldsymbol{w}$, which gives $(x\boldsymbol{I} - \boldsymbol{U}_0^+\boldsymbol{U}_1)\boldsymbol{w} = 0$, where $\boldsymbol{U}_0^+$ is the left pseudoinverse of $\boldsymbol{U}_0$. We conclude that the optimal x is an eigenvalue of $\boldsymbol{U}_0^+\boldsymbol{U}_1$ and, similarly, the optimal y is an eigenvalue of $\boldsymbol{U}_0^+\boldsymbol{U}_2$. Putting $\lambda = x + jy$ and computing $\hat{\rho} = \sigma_{min}([\boldsymbol{A} - \lambda\boldsymbol{I}\ \boldsymbol{B}])$, we obtain an upper bound for the controllability radius. If this bound is equal to the computed radius, then we are sure to have obtained the exact radius.

Comment 2. In both problems (8) and (19), the matrix variable is $O(n) \times O(n)$ and the number of scalar constraints (or free variables in $\boldsymbol{Q}$, if we use (10)) is $O(n^2)$. This leads to a complexity of $O(n^6)$ for solving the problems [13], using typical SDP algorithms as those from the library SeDuMi [14]. Customized algorithms, as those in [15], may achieve a reduction of this complexity. Despite an asymptotic complexity not better than that from [5] and [7], our experiments detailed in Section V show that our algorithms have good complexity for practical values of n.

Comment 3. In terms of accuracy, our approach may have difficulties when the ratio between the largest and the smallest singular value of $\boldsymbol{A}$ is large. The ratio between the largest and the smallest eigenvalue of $\boldsymbol{A}\boldsymbol{A}^H$ is the square of this large value. This may raise accuracy problems in assessing the positivity of $\boldsymbol{P}(\lambda) - \tau\boldsymbol{I}$. However, numerical experiments show that our algorithms fail relatively seldom.

III. Stabilizability radius

The stabilizability radius is defined similarly to (2), i.e. it is the norm of the smallest perturbation that makes the system unstabilizable. Its value is given by a smallest singular value minimization as in (3), but now with $\mathrm{Re}\lambda \geq 0$ for continuous-time systems and $|\lambda| \geq 1$ for discrete-time systems.

The stabilizability radius can be computed by a positive polynomial approach, as in the previous section, using recent results regarding matrix polynomials that are positive on bounded semialgebraic sets [16] (this is a generalization of the scalar version from [17]). Using real polynomials for the continuous-time stabilizability radius, we obtain a problem similar to (8), in which the constraint is $\boldsymbol{P}(x,y) - \tau \boldsymbol{I} \succeq 0$ on the set described by the inequalities $x \geq 0$ (to take the right half of the complex plane) and $r_2^2 - x^2 - y^2 \geq 0$ (to bound the set as prescribed in [16]). Then, the polynomial $\boldsymbol{P}(x,y) - \tau \boldsymbol{I}$ is positive if and only if it can be expressed as

$$\boldsymbol{P}(x,y) - \tau \boldsymbol{I} = \boldsymbol{S}_0(x,y) + x\boldsymbol{S}_1(x,y) + (r_2^2 - x^2 - y^2)\boldsymbol{S}_2(x,y), \tag{21}$$

where $\boldsymbol{S}_i(x,y)$, $i = 0:2$, are sum-of-squares matrix polynomials, parameterized as in (9), whose degree may be arbitrarily large. (Note that the polynomials multiplying the sum-of-squares are exactly those describing the positivity set.) Practically, we can take $\boldsymbol{S}_0(x,y)$ of degree 4 and $\boldsymbol{S}_1(x,y)$, $\boldsymbol{S}_2(x,y)$ of degree 2. (Note that taking $\boldsymbol{S}_0(x,y)$ of degree 2 would imply that the other two sum-of-squares are constants, which is fairly restrictive.) For the discrete-time case, the inequality $x \geq 0$ is replaced by $x^2 + y^2 - 1 \geq 0$ and (21) is changed accordingly. In both continuous- and discrete-time cases, the complexity of the SDP problem is still $O(n^6)$, but larger than for the computation of the controllability radius.

The results from [16] can be easily adapted to trigonometric polynomials, using the transformations described in [18] for the scalar case. The advantage of trigonometric polynomials is that boundedness always holds. Using the transformation (13), the right half of the complex plane is described by $\omega \in [-\pi/2, \pi/2]$, or equivalently, by $\cos\omega = (z_1 + z_1^{-1})/2 \geq 0$. Then, the inequality (16) holds on the right half plane if and only if

$$\boldsymbol{P}(z_1,z_2) - \tau \boldsymbol{I} = \boldsymbol{S}_0(z_1,z_2) + \frac{1}{2}(z_1 + z_1^{-1})\boldsymbol{S}_1(z_1,z_2), \tag{22}$$

where now $\boldsymbol{S}_i(z_1,z_2)$, $i = 0:1$, are trigonometric sum-of-squares. Maximizing τ subject to (22) gives the continuous-time stabilizability radius. Again, the degrees of the sum-of-squares may be arbitrarily large. However, in this case it appears that taking minimal values for the degrees of the sum-of-squares (i.e. degree $(1,2)$ for $\boldsymbol{S}_0(z_1,z_2)$ and $(0,2)$ for $\boldsymbol{S}_1(z_1,z_2)$) leaves enough freedom to obtain good approximations.

Computing the discrete-time stabilizability radius is even simpler. We only have to put $r_1 = 1$ in (13), (15), thus enforcing $|\lambda| \geq 1$, and then solve (19) with the new polynomial $\boldsymbol{P}(z_1,z_2)$.

So, with trigonometric polynomials, the stabilizability radius computation has about the same complexity as that of the controllability radius.

IV. Structured controllability radius

The positive polynomial approach can be extended to situations in which the state-space matrices depend polynomially on a (small) number of parameters. We consider here only the simplest case, the extensions being straightforward. Let us assume that the state transition matrix $\boldsymbol{A}$ is replaced with $\boldsymbol{A} + \alpha \boldsymbol{D}$, where the matrices $\boldsymbol{A}$, $\boldsymbol{D}$ are given and $\alpha \in [-1, 1]$ is an unknown parameter. We want to estimate the worst-case controllability radius, which means the smallest value of (3) over all the admissible values of the parameter α. In other words, we want to find the "less controllable" pair of the family $(\boldsymbol{A} + \alpha \boldsymbol{D}, \boldsymbol{B})$. The resulting problem is

$$\rho_{\boldsymbol{D}}(\boldsymbol{A}, \boldsymbol{B}) = \min_{\lambda \in \mathbb{C}, \alpha \in [-1,1]} \sigma_{min}([\boldsymbol{A} + \alpha \boldsymbol{D} - \lambda \boldsymbol{I} \ \boldsymbol{B}]). \tag{23}$$

Using the techniques described in Section II, this problem can also be solved using sum-of-squares polynomials. We discuss only the case of trigonometric polynomials, which is simpler. We put $\boldsymbol{M} = [\boldsymbol{A} + \alpha \boldsymbol{D} - \lambda \boldsymbol{I} \ \boldsymbol{B}]$ in (4). Instead of (5), we now have

$$\begin{aligned} \boldsymbol{P}(\lambda, \alpha) = |\lambda|^2 \boldsymbol{I} - \lambda \boldsymbol{A}^H - \overline{\lambda} \boldsymbol{A} + \boldsymbol{A}\boldsymbol{A}^H + \boldsymbol{B}\boldsymbol{B}^H \\ + \alpha^2 \boldsymbol{D}\boldsymbol{D}^H + \alpha(\boldsymbol{D}\boldsymbol{A}^H + \boldsymbol{A}\boldsymbol{D}^H - \lambda \boldsymbol{D}^H - \overline{\lambda} \boldsymbol{D}). \end{aligned} \tag{24}$$

By using the substitutions (13) and $\alpha = (z_3 + z_3^{-1})/2$, we obtain a trigonometric polynomial $\boldsymbol{P}(z_1, z_2, z_3)$ whose smallest eigenvalue can be minimized over the unit circle by solving a problem similar to (19), but now with three variables.

The only difficulty here is that the bounds r_1, r_2 have not to depend on α, as it would result by replacing $\boldsymbol{A}$ with $\boldsymbol{A} + \alpha \boldsymbol{D}$ in (12). This is easily fixed by computing overall bounds, using again positive polynomials. For example, for computing an upper bound for $\mathrm{Re}\lambda$, we have to solve the problem

$$\max_{\alpha \in [-1,1]} \lambda_{max} \left(\frac{\boldsymbol{A}+\boldsymbol{A}^H}{2} + \alpha \left(\frac{\boldsymbol{D}+\boldsymbol{D}^H}{2} \right) \right), \tag{25}$$

which is equivalent to

$$\begin{aligned} \min_{\beta} \quad & \beta \\ \text{s.t.} \quad & \beta \boldsymbol{I} - \left(\frac{\boldsymbol{A}+\boldsymbol{A}^H}{2} \right) - \frac{z+z^{-1}}{2} \left(\frac{\boldsymbol{D}+\boldsymbol{D}^H}{2} \right) \succeq 0, \forall |z| = 1 \end{aligned} \tag{26}$$

This is an optimization problem involving a positive matrix polynomial and, since the polynomial depends on a single variable, it can be solved exactly (it involves no relaxation), see e.g. [19]. Its complexity is negligible with respect to that of the structured controllability radius.

V. Experimental results

We have solved the SDP problems described in this paper using the library SeDuMi [14]. We call *d2ur* and *d2ut* the programs solving the controllability radius problems (8) and (19), respectively ('*d2u*' comes from 'distance to uncontrollability', while '*r*' and '*t*' indicate the type of polynomial, real or trigonometric). The programs have been run on a dual-core PC at 2.33 GHz, with 2 GB of memory. The programs are available at http://schur.pub.ro/Idei2007/d2u.zip.

For comparison with the methods from [5], [7], we have run our programs on 39 pairs $(\boldsymbol{A}, \boldsymbol{B})$ described in Table 4.1 of [7]. Both methods [5], [7] provide an interval in which the actual controllability radius should lie (this is the trisection version proposed in [6]). The method from [5] is the most accurate and we assume that all its results are correct. The method from [7] fails to give a correct interval for five pairs (Companion(10,4), Demmel(5,2), Demmel(10,4), Gallery(5,2), Godunov(7,3)). We note that the matrices $\boldsymbol{A}$ of these pairs have rather ill-conditioned eigenvalues and ratios $\sigma_{max}(\boldsymbol{A})/\sigma_{min}(\boldsymbol{A})$ going to 10^{18}. Our program *d2ur* fails to provide a controllability radius inside the correct interval only in two cases (Godunov(7,3) and Skew-Laplacian(8,3)), where it gives slightly larger values (so, the cause is numerical accuracy, not the sum-of-squares relaxation). The program *d2ut* gives a wrong result in four cases (Companion(10,4), Demmel(5,2), Demmel(10,4), Gallery(5,2)), where the computed τ_2 is negative, which again indicates numerical problems.

We have obtained a valid optimality certificate $\hat{\rho}$ (as described in Comment 1) in all tests where *d2ur* gave a correct result. The values *d2ur*$(\boldsymbol{A}, \boldsymbol{B})$, *d2ut*$(\boldsymbol{A}, \boldsymbol{B})$ and $\hat{\rho}$ coincide in the first 6 significant digits, in the cases where both *d2ur* and *d2ut* do not fail; *d2ut* is more accurate in these cases and gives always a radius smaller than $\hat{\rho}$, as expected. In 13 cases, we obtained *d2ur*$(\boldsymbol{A}, \boldsymbol{B}) > \hat{\rho}$.

We conclude that, although our algorithms may fail, they usually provide a fairly good accuracy. This conclusion is supported also by tests on normally distributed random matrices; we report here the results on 1000 pairs with $n = 5$, $m = 1$. Using the implementation of Gu's algorithm [5] from http://www.cs.nyu.edu/faculty/overton/software, we have computed intervals $(\rho_\ell, \rho_u]$ with a length of about 10^{-6}, in which the controllability radius lies. We have obtained $\rho_\ell \leq$ *d2ut*$(\boldsymbol{A}, \boldsymbol{B}) \leq \rho_u$ in 999 cases. In one case, where $\rho_u = 4.486 \cdot 10^{-5}$ (a very small value), the value returned by *d2ut* was $4.487 \cdot 10^{-5}$, i.e. very close to the upper bound. With *d2ur*, in 898 cases the computed radius was in $(\rho_\ell, \rho_u]$ and in 102 cases the radius was greater than ρ_u,

TABLE I

EXECUTION TIMES (IN SECONDS) OF CONTROLLABILITY RADIUS COMPUTATION PROGRAMS.

n	*d2ur*	*d2ut*	[5]	[7]
10	1.6	9.5	8.8	*171*
20	25	379	400	*881*
30	240	–	4196	*3003*
40	1103	–	–	*7891*

but typically very close to it. In a single case the error was significant, the same detailed above, where *d2ur* returned $2.26 \cdot 10^{-4}$.

The execution times of the programs are shown in Table I, with matrices $\boldsymbol{A}$ of type Kahan(n) and $\boldsymbol{B}$ generated randomly. The times for the algorithm from [7] were taken from that paper (that is why they are written in italics); we estimate that on our computer, the execution should be at most 10 times faster. Dashes in the table indicate that there was not enough memory to run the algorithm. We remark that, despite its $O(n^6)$ complexity, *d2ur* is faster than the other algorithms for a large range of matrix sizes. However, *d2ut* competes well in terms of speed only for small sizes; it also requires significantly more memory than *d2ur*; we could run it only for $n \leq 25$.

Example. We discuss now a simple example, going through all computations described in this paper. The pair is ([3], Example 2)

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0.1 & 3 & 5 \\ 0 & -1 & -1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 \\ 0.1 \\ 0 \end{bmatrix}. \tag{27}$$

The controllability radius computed by *d2ur* is 0.039238453, while *d2ut* gives 0.039238430. This is in accord with the result 0.039238 reported in [3]. Using the method described in Comment 1, we obtain $\hat{\rho} = 0.039238431$, which indicates that the result of *d2ut* is more accurate (and smaller than $\hat{\rho}$, as it should be).

The stabilizability radius is computed with the trigonometric polynomials approach presented in Section III. We name the programs *d2sc* for the continuous-time radius and *d2sd* for the discrete-time radius. We obtain *d2sc*$(\boldsymbol{A}, \boldsymbol{B}) = 0.039238444$ and *d2sd*$(\boldsymbol{A}, \boldsymbol{B}) = 0.039238430$.

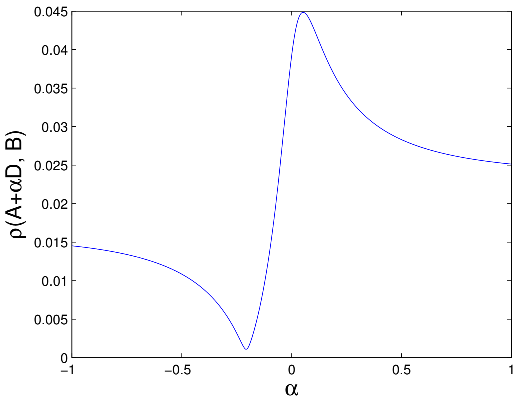

Fig. 1. Controllability radius $\rho(\boldsymbol{A} + \alpha\boldsymbol{D}, \boldsymbol{B})$, for $\alpha \in [-1, 1]$, for the numerical example (27), (28).

These are virtually the same values as for the controllability radius. Of course, this is not true in general. For example, while $\rho(\boldsymbol{A}, \boldsymbol{B}) = \rho(-\boldsymbol{A}, \boldsymbol{B})$, we obtain $d2sc(-\boldsymbol{A}, \boldsymbol{B}) = 0.3258033$.

Finally, we compute the structured controllability radius (23). We take

$$\boldsymbol{D} = \begin{bmatrix} -14 & -2 & 0 \\ 0 & 0 & 0 \\ -0.5 & -0.1 & 0 \end{bmatrix}. \tag{28}$$

Solving the corresponding trigonometric sum-of-squares relaxation, we obtain $\rho_{\boldsymbol{D}}(\boldsymbol{A}, \boldsymbol{B}) = 0.0010893$. The values $\rho(\boldsymbol{A} + \alpha\boldsymbol{D}, \boldsymbol{B})$, computed with *d2ut* for $\alpha \in [-1, 1]$, are shown in Figure 1; the minimum value (on a fine grid around the minimum) is 0.0010910, very close to the computed $\rho_{\boldsymbol{D}}(\boldsymbol{A}, \boldsymbol{B})$.

VI. Conclusions

We have proposed SDP problems that compute the controllability radius (3) of a linear system, using properties of sum-of-squares polynomials. In terms of complexity and accuracy, our algorithms occupy a distinct place among successful methods as those from [5], [6], [7]. Our approach has also the strength of flexibility, as shown by the structured problem treated in section IV.

Further work will be devoted to extend our algorithms to other controllability radius problems, for example to the case of descriptor or higher order systems.

Acknowledgment. We thank Dr. Emre Mengi for providing the data used in [7].

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