LOW-COMPLEXITY ROBUST DOA ESTIMATION

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ABSTRACT

We propose a low complexity method for estimating direction of arrival (DOA) when the positions of the array sensors are affected by errors with known magnitude bound. This robust DOA method is based on solving an optimization problem whose solution is obtained in two stages. First, the problem is relaxed and the corresponding power estimation has an expression similar to that of standard beamforming. If the relaxed solution does not satisfy the magnitude bound, an approximation is made by projection. Unlike other robust DOA methods, no eigenvalue decomposition is necessary and the complexity is similar to that of MVDR. For low and medium SNR, the proposed method competes well with more complex methods and is clearly better than MVDR.

***Index Terms*—** direction of arrival, spectral power density, robustness, optimization

1. INTRODUCTION

Direction of arrival (DOA) estimation is a core problem in array signal processing, with numerous applications. One of its facets, to which we contribute in this paper, is that of robustness to errors in sensor positions, due to improper calibration or to inherent position uncertainty, like in the case of an array towed by an underwater unmanned vehicle (UUV). There are at least two general ways of enforcing robustness.

Worst-case robustness ensures a minimum beamforming quality for all the steering vectors in a neighborhood $\mathcal{V}$ of the nominal vector that corresponds to the correct position. Methods belonging to this category can be found, among others, in [1, 2, 3]. In [4], beamforming methods based on convex optimization are reviewed, including worst-case designs.

A second way is to find the steering vector in $\mathcal{V}$ that maximizes the estimation of the power on the current direction, for the data at hand. Examples of such approaches are [5, 6]. To other categories belong methods like the maximally robust Capon beamformer [7]. The recent review [8] lists several other methods.

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Most of the robust DOA methods are computationally heavy, usually much more complex than the standard MVDR. On the other hand, MVDR may give quite wrong estimations in the presence of sensor errors.

Contribution and relation to prior work. We present a method that aims to provide robustness at a cost similar to that of MVDR and thus be appropriate to the computation power of a simple UUV. Our method belongs to the second class, being in the direct line of [5, 6], but its complexity is significantly lower; in particular, no eigenvalue decompositions are required. The innovation comes from the form of the optimization problem, which allows a simple but meaningful solution approximation.

2. OPTIMIZATION PROBLEM

DOA problem. We study first the case of narrowband sources and, for the sake of simplicity, consider that the sources and the array sensors are in the same plane. The steering vectors associated with the array have the expression

$$\bar{\boldsymbol{a}}(\theta) = \begin{bmatrix} \dots & e^{-j2\pi f \tau_k(\theta)} & \dots \end{bmatrix}^T \tag{1}$$

where $\tau_k(\theta)$ is the delay of the signal of frequency f coming from direction θ, at the k-th sensor; the array has N sensors. The delay, measured with respect to a reference sensor, depends on the position of the sensor (and on the propagation speed). In what follows we will omit the angle θ in the notation.

MVDR. An often used solution for estimating the power of the signal is the MVDR estimation, found by solving (for each angle) the optimization problem

$$\begin{array}{ll} \min_{\boldsymbol{w}} & \boldsymbol{w}^H \boldsymbol{R} \boldsymbol{w} \\ \text{s.t.} & \boldsymbol{w}^H \bar{\boldsymbol{a}} = 1 \end{array} \tag{2}$$

where $\boldsymbol{w}$ is the weights vector for the sensor signals and $\boldsymbol{R}$ is the covariance matrix of these signals. The power on the angle defining $\bar{\boldsymbol{a}}$ has the simple analytic expression

$$\sigma^2 = \frac{1}{\bar{\boldsymbol{a}}^H \boldsymbol{R}^{-1} \bar{\boldsymbol{a}}}. \tag{3}$$

In practice, the ideal covariance matrix $\boldsymbol{R}$ is replaced by its estimation based on the array signal $\boldsymbol{x}_t$ measured over T samples

$$\hat{\boldsymbol{R}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{x}_t \boldsymbol{x}_t^H. \tag{4}$$

Robustness. The steering vectors may have not their ideal values (1) due to, for example, sensor displacement with respect to their assumed nominal positions. In this case, the estimation (3) may be far from the correct value. A typical way to cope with this situation is to include robustness to uncertainties or possible errors of the steering vector into the optimization problem. A successful approach is to solve [5]

$$\begin{array}{ll} \min_{\boldsymbol{a}} & \boldsymbol{a}^H \boldsymbol{R}^{-1} \boldsymbol{a} \\ \text{s.t.} & \|\boldsymbol{a} - \bar{\boldsymbol{a}}\|^2 \le \rho \end{array} \tag{5}$$

where ρ is a given constant bounding the distance between the actual and the nominal steering vectors. One can view (5) as maximizing the estimated spectrum (3) using steering vectors in a neighborhood of the nominal one.

Since for an $\boldsymbol{a}$ with smaller magnitude (but constant direction), the objective of (5) is smaller, the solution lies on a hypersphere around the tip of $\bar{\boldsymbol{a}}$. More precisely, on a part of the hypersphere whose maximum extent is limited by $\|\boldsymbol{a}\| \le \|\bar{\boldsymbol{a}}\|$. A simplified illustration is given in Figure 1, where the possible positions of the solution are on the red arc. So, the vectors with smaller norm, whose direction is close to $\bar{\boldsymbol{a}}$, are favored. Normalization, which is a natural remedy, partly cures this intrinsic bias towards the nominal position, but this is only an a posteriori cure. Using a Lagrange multipliers approach, the solution of (5) can be found relatively quickly, by finding the unique zero of a simple function; however, the full eigenvalue decomposition of $\boldsymbol{R}$ is necessary.

If the errors are only in the positions of the sensors, then the steering vectors errors appear in the phase, while the norm of the steering vectors is unchanged $\|\boldsymbol{a}\|^2 = \|\bar{\boldsymbol{a}}\|^2 = N$. In this case, the robustness problem can be better approximated with

$$\begin{array}{ll} \min_{\boldsymbol{a}} & \boldsymbol{a}^H \boldsymbol{R}^{-1} \boldsymbol{a} \\ \text{s.t.} & \|\boldsymbol{a} - \bar{\boldsymbol{a}}\|^2 \le \rho \\ & \|\boldsymbol{a}\|^2 = N \end{array} \tag{6}$$

The solution is known as the doubly constrained Capon beamformer [6] and can again be computed using the Lagrange multiplier approach and the full eigenvalue decomposition, although the problem is not convex. The feasible domain of (6) is illustrated by the green arc in Fig. 1.

Proposed approach. Still assuming only phase errors in (1), we propose to pose the problem differently, by using a flat surface (a hyperplane) orthogonal on $\bar{\boldsymbol{a}}$ instead of the (convex) hypersphere in (5) or the (nonconvex) equinorm surface from (6). This ensures a lesser impact of the vectors norm on the criterion, compared to (5), while preserving the convex character of the optimization. Compared to (6), this is a relaxation; we will show however that it has some advantages.

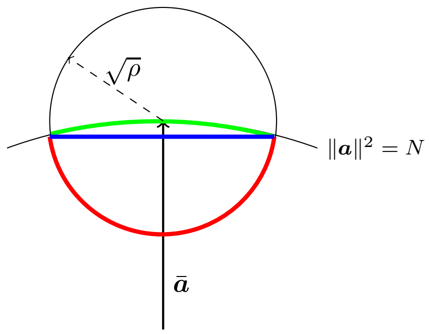

Fig. 1. Possible positions of the optimal steering vector in problems (5) (red), (6) (green) and (7) (blue).

In Figure 1, the possible domain is illustrated by the blue line. The problem to be solved is

$$\begin{array}{ll} \min_{\boldsymbol{a}} & \boldsymbol{a}^H \boldsymbol{R}^{-1} \boldsymbol{a} \\ \text{s.t.} & \boldsymbol{a}^H \bar{\boldsymbol{a}} = \gamma N \\ & \|\boldsymbol{a} - \bar{\boldsymbol{a}}\|^2 \le \rho \end{array} \tag{7}$$

The scalar γ is chosen such that the intersection of the hyperplane with the hypersphere is a lower dimensional hypersphere of radius $\sqrt{\rho}$. Simple geometry gives $\gamma = 1 - \rho/2N$.

The problem (7) appears more complicated than (5), but we will show that its exact solution can be computed with about the same effort and we will also present a simplified solution.

Proposed algorithm. The first step is to solve (7) without the second constraint. The solution is

$$\boldsymbol{a} = \frac{\gamma N \boldsymbol{R}\bar{\boldsymbol{a}}}{\bar{\boldsymbol{a}}^H \boldsymbol{R}\bar{\boldsymbol{a}}}. \tag{8}$$

The corresponding power estimation (the optimal value of (7)) is

$$\sigma^2 = \frac{\bar{\boldsymbol{a}}^H \boldsymbol{R}\bar{\boldsymbol{a}}}{\gamma^2 N^2}. \tag{9}$$

As advocated in [5], normalization of the steering vector produces a better estimate, given by

$$\hat{\sigma}^2 = \sigma^2 \frac{N}{\|\boldsymbol{a}\|^2}. \tag{10}$$

If $\|\boldsymbol{a}\|^2 \le N$ (or, equivalently, $\|\boldsymbol{a} - \bar{\boldsymbol{a}}\|^2 \le \rho$), then (8) is also the solution of the full problem (7). The numerator of the power estimate (9) has in fact a familiar form, that resulting from standard beamforming. So, we could call *adjusted beamforming* the proposed robust estimator. Note also that its computation implies at most matrix-vector multiplications.

If $\|\boldsymbol{a}\|^2 > N$, then (8) is not a valid solution. However, due to convexity, the solution of (7) is the same as that of the problem

$$\begin{array}{ll} \min_{\boldsymbol{a}} & \boldsymbol{a}^H \boldsymbol{R}^{-1} \boldsymbol{a} \\ \text{s.t.} & \boldsymbol{a}^H \bar{\boldsymbol{a}} = \gamma N \\ & \|\boldsymbol{a}\|^2 = N \end{array} \tag{11}$$

The difference is that the second constraint is now an equality. A solution to this type of problem was given in [6], for norm-constrained beamforming, with different meaning for the variable (the weights vector there) and different significance. However, the algorithm applies here also. Similarly to solving (5), it implies the eigenvalue decomposition of $\boldsymbol{R}$ and finding the zero of a function of a single variable. So, the complexities of (5), (6) and (11) are essentially the same.

Simplified version. It may be interesting in practice to seek a simple approximate solution to (11), taking advantage of the already computed (8). Denote

$$\boldsymbol{y} = \boldsymbol{a} - \gamma \bar{\boldsymbol{a}} \tag{12}$$

and note that $\boldsymbol{y}^H \bar{\boldsymbol{a}} = 0$. We propose to replace the unvalid $\boldsymbol{a}$ with an $\hat{\boldsymbol{a}}$ that satisfies the constraints of (11) and lies on the same plane as $\boldsymbol{a}$ and $\bar{\boldsymbol{a}}$, between these two vectors. This is not the nearest vector from $\boldsymbol{a}$ satisfying the constraints, but can be seen as the projection of $\boldsymbol{a}$ on the constraints in the direction of $\bar{\boldsymbol{a}}$. Using orthogonality, it results that the approximate solution is

$$\hat{\boldsymbol{a}} = \gamma \bar{\boldsymbol{a}} + \sqrt{N(1-\gamma^2)} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}. \tag{13}$$

Its computation comes at almost no extra cost with respect to (8). For the power estimation we use (3) with $\hat{\boldsymbol{a}}$ replacing $\bar{\boldsymbol{a}}$, noting that $\|\hat{\boldsymbol{a}}\|^2 = N$. We will show in the next section that this simple approximation, although suboptimal, is useful under certain conditions that are fairly common in practice.

Wideband case. When the sources have a wideband spectrum, as typical in passive DOA, we employ two standard techniques described in [9]. For both, the sensor signals are decomposed via DFT on frequency bands. The summation method simply builds covariance matrices (4) for each frequency band, independently estimates the powers for the desired angles and then sums (or averages) the results. The coherent transformations method uses diagonal focusing matrices that, for each frequency band and angle, project the sensor signals on a single central band; then, a unique covariance matrix is obtained and the preferred narrowband method can be applied.

An important issue is the choice of the robustness parameter ρ, for each frequency band in the summation method and for the central band in the transformation method. Since in a first order approximation, the difference between a perturbed steering vector and the nominal one (1) depends linearly on the position error, we propose that the robust methods receive a single value of ρ corresponding to the Nyquist frequency. For the other bands, this value is multiplied with the ratio between the corresponding frequency and the Nyquist frequency.

Regarding complexity, we note that the methods based on (5) or (6) need a full eigenvalue decomposition for each band, in the summation method, and for each angle, in the transformation method. Our simplified method is free of eigenvalue decompositions and has complexity similar to that of MVDR.

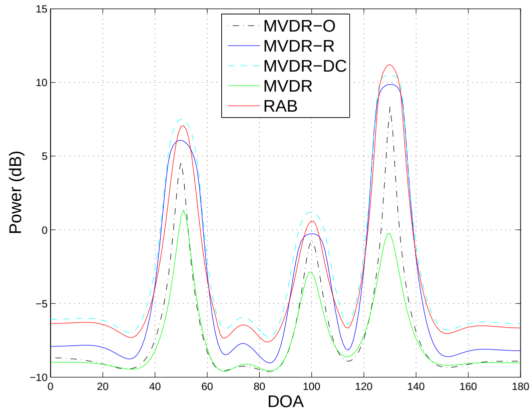

Fig. 2. Narrowband, SNR $= 10$ dB, simplified RAB.

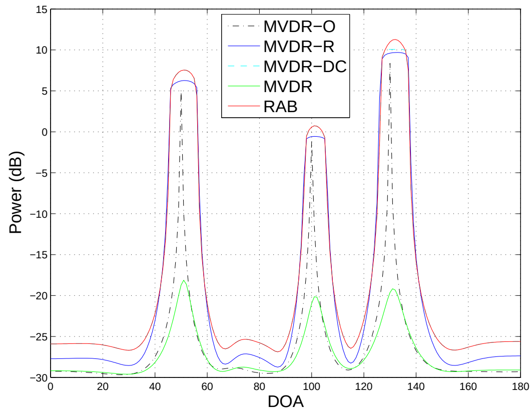

Fig. 3. Narrowband, SNR $= 30$ dB, exact RAB.

3. EXAMPLES

For testing our algorithm we used simulations for both narrow- and wideband cases. We consider a uniform linear array (ULA) with $N = 12$ sensors; the distance between two consecutive sensors is $d = 0.25$ m. We assume an underwater DOA problem, the speed of sound being 1450 m/s. The sampling frequency is 8 KHz.

Since we assume that the actual sensor positions are uncertain, we perturbe the nominal ULA positions with white Gaussian noise with standard deviation $\delta = 0.2d$, on both coordinates. Since the delay at a sensor is changed by δ/c for a displacement of size δ in the worst-case, when the displacement is in the direction of a source, the modification of the squared norm of the steering vector (1) is about $0.18N$. We chose the covering value $\rho = 0.3N$ for the robustness parameter in problems (5), (6) and (7).

The signals at sensors are simulated using the true sensor positions. White noise is added such that a desired SNR is obtained. Note that the SNR does not account for the perturbed sensor positions, which actually decrease even more

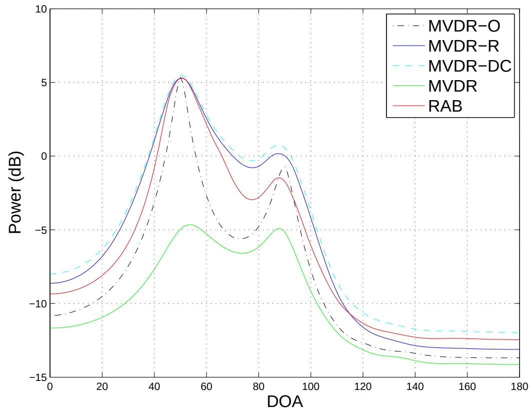

Fig. 4. Wideband, SNR $= 10$ dB, simplified RAB.

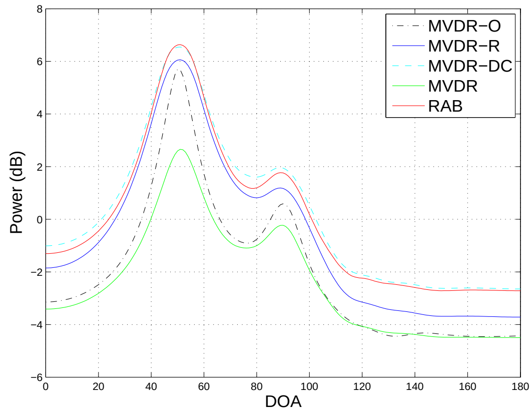

Fig. 5. Wideband, SNR $= 0$ dB, simplified RAB.

the quality of the signals.

Besides our method, called RAB (robust adjusted beamforming), we used in simulations the standard MVDR, the robust method based on (5), denoted here MVDR-R, and the doubly constrained method solving (6), denoted MVDR-DC. All these methods use the nominal steering vectors, i.e. have no knowledge of the position perturbations. For comparison, we report the results of the "oracle" MVDR, denoted MVDR-O, which uses the actual steering vectors, based on the true sensor positions; these are the best results that can be obtained with a low-complexity method. We call *exact* RAB our method based on (8)–(11) and *simplified* RAB that using (8)–(10) and (13).

Narrowband case. We simulate three sources, whose DOAs are 50, 100 and 130 degrees and whose amplitudes are 2, 1, 3, respectively. Their frequency is 2000 Hz.

The simplified RAB has the following typical behavior. In the vicinity of the true DOAs, the solution (8)–(10) is valid, hence there is no need for further search. In the other, less relevant directions, the solution is no longer valid, that is, $\|\boldsymbol{a}\|^2 > N$ for (8); however, the approximation (13) is good enough. This behavior is especially seen for medium and low SNR and an example is shown in Figure 2, for an SNR of 10 dB. One can see that simplified RAB, MVDR-R and MVDR-DC give quite similar estimations, consistent with those of MVDR-O, despite the significantly lower complexity of RAB. The standard MVDR can still give fairly good DOA estimation, but the powers of the sources are badly estimated. For lower SNR this relative behavior is somewhat similar, although the differences between the methods are smaller, due to the prevalence of the sensor measurement noise over the "noise" due to unknown sensor positions.

At higher SNR (e.g. more than 20 dB), the simplified RAB starts missing some peaks of the power, especially when the approximation (13) has to be used, and becomes less reliable although still significantly better than standard MVDR. In this situation it is recommended to use the exact RAB, whose results are illustrated in Figure 3, where the SNR is 30 dB. As expected, the estimation of exact RAB is similar with that of MVDR-R, with the slight advantage of more pointed peaks instead of the rather flat ones typically given by MVDR-R. The results of exact RAB and MVDR-DC almost coincide, with the typical exception of the highest peak, where MVDR-DC is perfectly flat; this behavior occurs because for angles near the strongest source, MVDR-DC tends to approximate the steering vector with the eigenvector of the covariance matrix (4) corresponding to the largest eigenvalue.

Wideband case. We simulated two sources corresponding to AR processes whose spectrum is decaying as frequency grows, as common for ships, see [10] for details. They are situated at angles 50 and 90 and their amplitudes are 2 and 1, respectively. We use an FFT of size 128 and $T = 62$ snapshots per frame, which means about one second per frame. We use the frequency bands from 500 to 3000 Hz. The central frequency for the transformation method is 2000 Hz.

The results of the summation and transformation methods are similar, so we report results only for the former. Figures 4 and 5 give examples of power estimation for a single frame, at SNRs of 10 and 0 dB, respectively. Again, the robust methods give better estimates of the relative amplitudes of the peaks and are close to MVDR-O. Simplified RAB appears quite reliable and gives good estimates at a cost significantly lower than MVDR-R or MVDR-DC.

4. CONCLUSIONS

We have presented a low complexity robust DOA estimation method, for arrays whose sensors positions are uncertain. The method is based on a new way of posing the optimization problem and on a simple but effective approximation. The simulations for narrowband and wideband sources show that the method gives good performance at low and medium SNR. Although the complexity is similar to that of MVDR, the estimation quality approaches that of more complex methods.

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